POWER LAW SPEECH COMPRESSION
AND EXPANSION

E. RAYMOND KNICKEL

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by

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PREFACE

This thesis covers a rather novel method for increasing the power conveying intelligence in a speech modulated radio telephone system. The basic idea for the method used was conceived while trying to find some means for reducing bandwidth requirements for speech transmission. This led to considering the possibility of electronically making a trigonometric transformation according to the trigonometric identity,

$$\sin \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{2}} \quad ,$$

in order to obtain an output frequency one half of the input frequency. It did not seem practical to use this idea for bandwidth compression, however, as among other things, it would require breaking up the modulating wave into all of its simple sinusoidal components at the point of transmission. In order to extract the intelligence, it would require a complicated inverse operation at the point of reception.

This led to the realization, however, that if one could perform some function, such as taking the square root of the input wave that there would result instantaneous compression. Furthermore, having performed a known function in the process of compression, one could theoretically reconstruct the input wave exactly by performing the inverse function in an expansion process at the point of reception.

Unfortunately, the mathematical analysis for the operation of this type of system had to be extremely limited. This results from

complications arising due to the non-linearities that are inherent to the system. The mathematics of non-linear systems is such that it seems impossible to develop a general method for analyzing this system. Therefore, the mathematical analysis had to be greatly simplified and even then, answers that reveal relatively small amount of information required much tedious mathematics.

Throughout this thesis, the analysis presumes that this system would find it's greatest use in rather simple radio communication systems such as aircraft, marine or amateur use.

The author wishes to express his appreciation to Assistant Professor C. F. Klamm, Jr. of the U S. Naval Postgraduate School for his many helpful suggestions in preparing the final copy of this paper.

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CHAPTER I

INTRODUCTION

Summary: This thesis presents an analysis of an instantaneous speech compression system that theoretically enables accurate reproduction of the original signal at the point of reception. The analysis is primarily based on a "half-power law" compression resulting in an effective power gain of about 7 db. The receiving system performs the inverse operation thereby reproducing the original signal. Perfect reproduction increases bandwidth requirements, but it is indicated that satisfactory reproduction can be obtained without exceeding normal bandwidth requirements. The probable effects of amplitude and phase distortion within the system are discussed. Circuits for accomplishing the indicated functions are developed and experimental data on these circuits is presented.

It is the purpose of this thesis to analyze a proposed method of increasing the average intelligence conveying power in a radio telephone system, particularly as it applies to voice modulated systems. It is proposed to employ as the modulating source for a radio telephone transmitter an audio compressing amplifier that will have an output instantaneously proportional to the square root of the absolute value of the modulating wave with the instantaneous input polarity being retained by the output. That is, when the input wave is positive, the output will be positive and proportional to the square root of the input wave, and when the input wave is negative, the output will be negative and proportional to the square root of the absolute magnitude of the input wave.

The receiver to be used with this system would be expected

to employ an audio expanding amplifier whose output would be proportional to the "square" of the absolute value of the instantaneous audio input voltage. Hence the receiver would act to remove the distortion introduced by the non-linearity of the amplifier employed at the transmitting end, and a true reproduction of the transmitted voice would be received.

The analysis also includes some data on power relations different from that of the square root. In other words, if E_{in} is the input audio wave, the output wave will be of the form $A(E_{in})^m$ where m takes on values different from 0.5. Throughout this thesis, m will refer to the value of the exponent involved. Where m is not specified, it will be assumed equal to 0.5.

This scheme will act to increase the average intelligence conveying power in two ways. First, in conversational speech, the ratio of 1/8 sec. peak to averaged power is roughly 12 db.[1] This indicates a ratio of 1/8 sec. peak to averaged modulating power of 16 to 1, or a ratio of 1/8 sec. peak to averaged voltage of 4 to 1. Hence, with conversational speech, the averaged percentage of modulation can only be a maximum of about 25%. For example, with a Class C plate modulated transmitter, the average peak modulating voltage equals 25% or 1/4 of the d.c. plate voltage and therefore the peak modulating current also equals 1/4 of the d.c. plate current. If the plate voltage is E_b and the plate current is I_b, then the peak modulating power is $1/4\ E_b \times 1/4\ I_b$ or $1/16\ E_b I_b$. Converting this to effective power, the effective modulating power is $1/4\ E_b \times 0.707 \times 1/4\ I_b \times 0.707 = 1/32\ E_b I_b$ or 1/32 of the d.c. power supplied to the plate circuit of

the Class C stage.

This means that a 1 kilowatt output transmitter actually yields only 31.2 watts of intelligence conveying power. By the same reasoning, a single sideband transmitter capable of 1 kilowatt power output, the actual average output during periods of modulation is about 62.5 watts.

By using some method of reducing the peak to averaged ratio of power in a speech wave, it is possible to increase the intelligence conveying power in an amplitude modulated system.

One of these methods is to use peak clipping and another method is the so-called "super modulation". Both of these methods introduce undesirable distortion in the system which cannot be removed by any simple means at the receiving end of the system.

The system proposed by this thesis also introduces distortion, but in a manner in which it can be eliminated at the receiving end, This automatically results, since a known function is to be performed on the modulating signal and the inverse of that function can therefore be accomplished at the receiving end, thereby eliminating any distortion that has been introduced at the transmitting end. With "peak clipping", or "super-modulation", the function being performed on the modulating wave varies with its instantaneous amplitude and hence the inverse of the function cannot be determined.

By taking the "square root" of the modulating voltage, we reduce the ratio of maximum peak amplitude to average peak amplitude of the audio wave from about 4 to 1 down to about $(4)^{1/2}$ to 1, or 2 to 1. Hence, the same audio signal can now be used to produce

an average percentage of modulation of 50%. This increases the amount of power conveying intelligence by a factor of four.

The second manner in which the proposed system increases the power conveying intelligence is by reason of the fact that the audio wave will now have a higher R.M.S. to peak ratio than before.

A sine wave has a R.M.S. to peak ratio of 0.707. If our modulating wave can be represented by a sine wave, the R.M.S. value now equals

$$\sqrt{\frac{\left(\int_0^{\frac{\pi}{2}} \sin\theta\right) d\theta}{\pi/2}} = \sqrt{\frac{\left[-\cos\theta\right]_0^{\frac{\pi}{2}}}{\pi/2}} \doteq 0.8$$

Thus the total increase in effective power will be by a factor of $2^2 \times \left(\frac{0.8}{707}\right)^2 = 5.1$, or about 7 db.

The advantages to be gained by this method from the standpoint of power are obvious. However, the advantages or disadvantages of this system from standpoint of bandwidth requirements are not so obvious.

For instance, if we passed a square wave having rounded edges through this amplifier, as shown by the solid line in Figure 1, the output waveform would be of the form indicated by the dashed line. Since the dashed line has sharper corners than the solid line, the energy contained in the higher order harmonies has been increased. Let us now take a waveform described by

$$y = \sin^2 \theta \qquad (2n+1)\pi \leq \theta \leq (2n+2)\pi$$

$$y = \sin^2 \theta \qquad (2n+2)\pi \leq 0 \leq (2n+3)\pi$$

giving us the waveform represented by the solid curve of Figure 2.

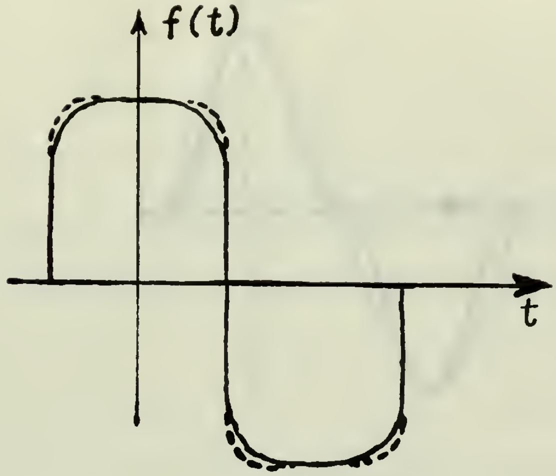


Figure 1

Waveform at output and input of compressing amplifier with a distorted square wave input wave form; Solid curve represents distorted square wave input. Dashed curve is the same waveform after passing through the compressing amplifier.

This waveform is obviously rich in harmonics, and yet when it is passed through the proposed amplifier, the output will be simply a plot of sin θ , as indicated by the dashed curve of Figure 2, which is certainly not rich in harmonics, but is a pure sine wave having no harmonic content. Thus, in one case, this amplifier has increased the harmonic content of a wave, and in another case it has decreased the harmonic component of a wave. It is apparent that whether the harmonic content of a wave is increased or decreased is a function of the nature of the input wave.

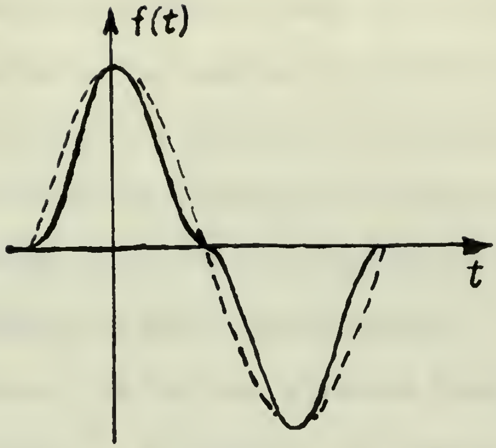


Figure 2.

Waveform at output and input of compressing amplifier with a sinusoidal output waveform; Solid curve is a plot of the function described by

$$y = \sin^2 \theta \qquad (2n+1)\pi \leq \theta \leq (2n+2)\pi$$

$$y = \sin^2 \theta \qquad (2n+2)\pi \leq 0 \leq (2n+3)\pi$$

Solid curve is the sine wave that results from passing this waveform through the compressing amplifier.

Since the nature of a speech wave is continually changing, one can anticipate that at times the audio amplifier employed at the transmitter will be increasing the number of harmonics in the wave, while the audio amplifier at the receiver will be decreasing the number of harmonics present and at other times the role played by the two amplifiers will, in this respect, be reversed.

To attempt to analyze mathematically the effect of this system on the bandwidth requirements for speech transmission seems an endless task that would yield results of questionable value. The bandwidth requirements will, of course, be determined by characteristics of the sending end audio system and the nature of the intelligence

being transmitted, which in this case will be some form of speech. The bandwidth requirements could hardly be expected to be the same as for speech transmission employing only linear amplifiers. Since, as was pointed out earlier, the sending end amplifier may act to either increase or decrease the frequency components of the input wave, it would theoretically be necessary to analyze the effect of the sending end audio system on every conceivable speech input wave, of which there are an infinite number, called phones. The speech sounds of any language, however, do fall into discrete intelligence conveying patterns called phonemes.[2] In English, the number of phonemes is approximately 48.[3] The basic frequencies in any phoneme, of course, vary from individual to individual, particularly between a male adult and a female adult, or an adult and a child. There is also some difference in the phoneme patters between individuals of the same sex and age group coming from different sections of the country.

If we were to restrict ourselves in the analysis to one person's voice, it might be possible to mathematically analyze the effect of this system on required bandwidth by picking several, say ten, instants of time during the space of every phoneme and then determining the increase or decrease in bandwidth for each such instant of time. This would reduce the problem to determining about 500 different speech waveforms. It would then be necessary to take the "square root" of each wave form so determined and then run a Fourier analysis on each resultant waveform. Obviously, even this simplified mathematical approach is far too time consuming to make a part of this thesis.

Fortunately, the phonemes which require the greatest bandwidths for proper transmission are the voiceless phonemes, of which there are about 8. The only exception to this seems to be in the case of the voiced fricative sound "z" as in "zoo".[4] Of this group, the phoneme "h" (as in "he") does not require an appreciable bandwidth. Since these phonemes are unvoiced, their natures should not differ greatly between indi viduals. Furthermore, there may be sufficient similarity between these sounds so that they will be affected in a similar manner so far as bandwidth requirements are concerned. However, even in these cases, it is felt that it is best to perform a laboratory type analysis first, and then let the results of such an analysis point the way to profitable areas of mathematical analysis.

CHAPTER II
CIRCUIT DESIGN

The problem of determining a circuit that will have an output that is instantaneously proportional to the square root of the input signal is one that apparently no one has yet neatly accomplished with a simple circuit.[5] Conventional so-called square law devices or circuits in general have a characteristic that only approximates a square law relationship. Furthermore, the approximate square law relationship will hold only over a limited range. It would appear that vacuum tubes will give the closest approximation to a power law relation with miniature copper oxide rectifiers and germanium diodes in that order giving the next most desirable characteristics.[6] Unfortunately, when using vacuum tubes, it is necessary to compensate for contact potential and the fact that a vacuum tube may draw some current with zero plate voltage. These compensating circuits complicate the device and could very well be a source of frequency distortion. Copper oxide rectifiers have characteristics that vary with temperature rather noticeably. They tend to introduce some frequency distortion, and it appears that it would be difficult to obtain values of m in the order of 0.5. Hence, it was decided to first attempt to obtain the desired exponential characteristics by using germanium diodes.

A personal contact with Mr. A. J. Hiller of the Naval Research Laboratory indicated that it should not only be possible to obtain the desired characteristics, but that it should be possible to alter the

characteristics slightly by varying the amount of resistance in a simple series resistance and germanium diode circuit.

The plot of a 1N34 crystal diode characteristic curve is a straight line on a log log plot over a large range of values. A straight line on log log paper may be represented by the equation $Y = aX_m$. Hence, the equation of the curve is

$$E = a I^m \qquad \text{Eq. (1)}$$

The circuit for indicating these characteri stics is shown in Figure 3.

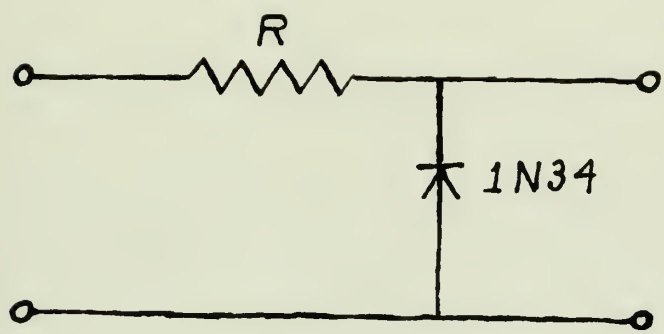


Figure 3

Circuit for obtaining characteristics of Equation 1.

If R is much greater than the resistance of the 1N34 diode, then

$$E_{out} = a I^m = a\left(\frac{E_{in}}{R}\right)^m \qquad \text{Eq. (2)}$$

Hence, the output voltage is proportional to some power, m, of the input voltage, giving us a function of the desired form.

In order to obtain symmetrical response on both excursions of the input wave, it is necessary to use a back-to-back arrangement

of the crystal diodes, as shown in Figure 4.

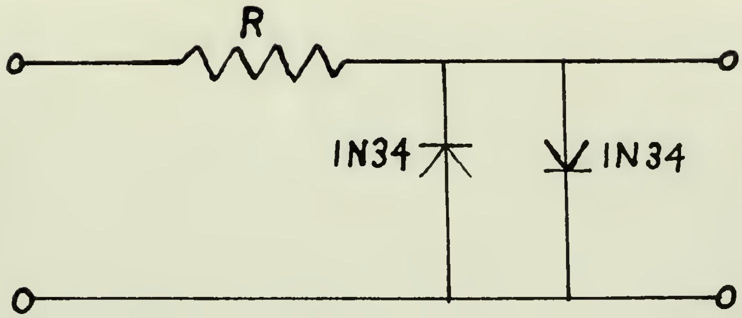


Figure 4

Circuit for obtaining symmetrical response corresponding to Equation 2.

It is necessary, of course, that the 1N34's be matched. The circuit for selecting matched crystals is shown in Figure 5.

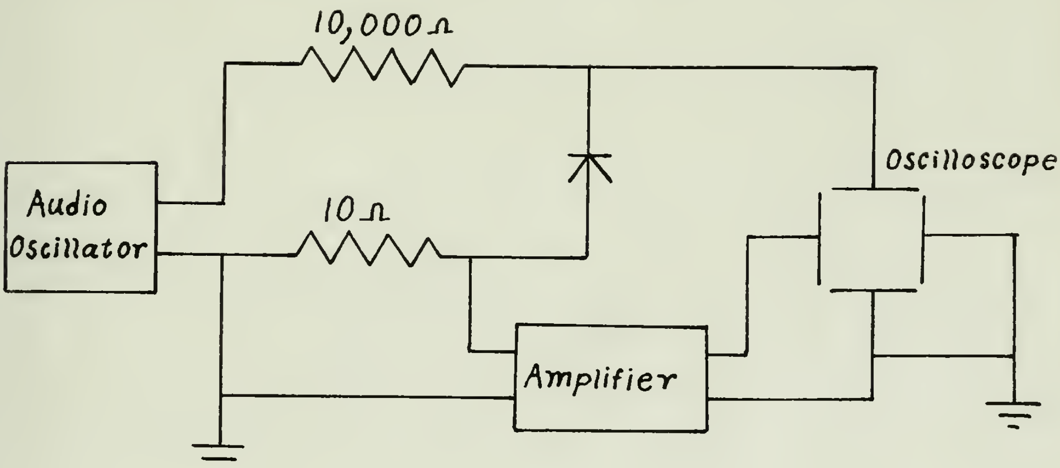


Figure 5.
Circuit for selecting matched crystal diodes

The voltage developed across the 10 ohm resistor is proportional to the current through it. Since 10 ohms is much lower than the resistance of the 1N34 diode at any point in its characteristic curve, the 10 ohm resistor does not appreciably affect the total amount of voltage between point "A" and ground. Thus, the characteristics of the 1N34 diode in series with the 10 ohm resistor are the same as for the 1N34 diode by itself.

Since the horizontal deflection voltage is proportional to the current through the 1N34 and the vertical deflection is proportional to the voltage impressed across the 1N34, a characteristic curve is displayed on the oscilloscope. Several diodes were then tried until two were found having an identical characteristic curve over the desired range. Several matched pairs were then found. It is, of course, unlikely that any two sets would have identical characteristics.

The diodes were then connected in an arrangement as shown in Figure 6 and the characteristics of this circuit plotted. Several values of R were tried. Best performance seemed to be obtained with $R = 10K$.

Using crystal set # 1, this gave a value of m that varied from 0.366 with $R = 0$ to $m = 0.577$ with $R = 500$ ohms.

In general, the curves follow a specific exponential relation to within 10% over about a 40 db range. Considering that the measuring equipment used had a maximum probable error of 3% of full scale, this is considered satisfactory. A log log plot of E_{out} vs. E_{in} is shown in Figure 7, 8, 9, 10, 11, and 12.

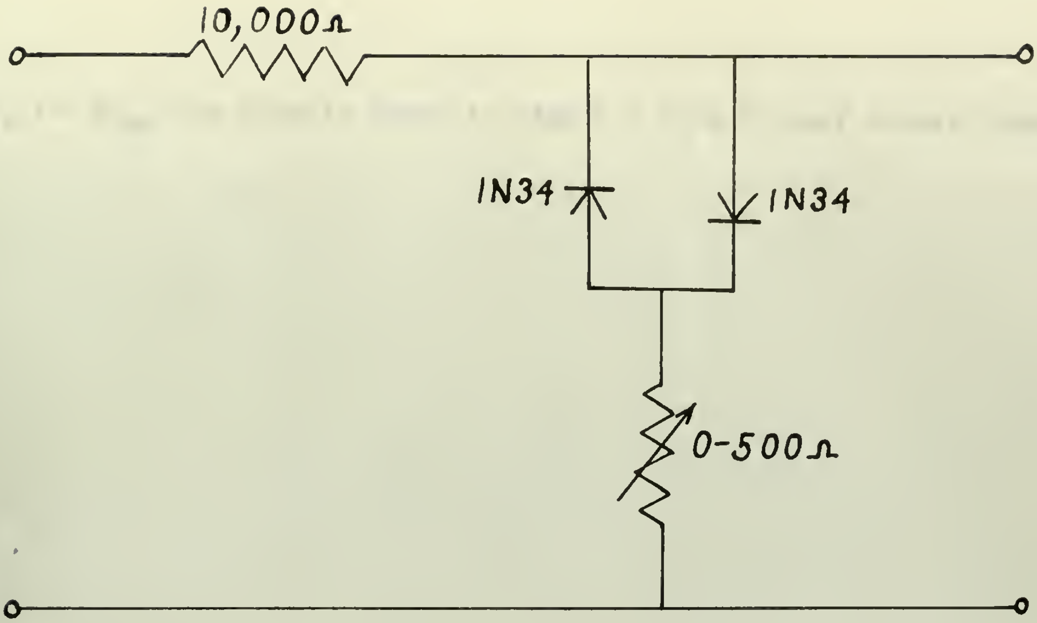


Figure 6

Circuit for obtaining a response corresponding to Equation 2 with m variable over a limited range.

As can be seen from the curves, it is necessary that an amplifier having the proper gain precede the unit in order that the speech signal will have the proper level for the operating range. The center of the operating range shifts slightly with different values of R .

Units of the type shown in Figure 6 have an insertion loss that remains constant at about 13 db as R is varied.

Figure 13 shows a plot of m vs. R.

The conclusions to be drawn regarding a circuit of the type shown in Figure 6 are that the circuit will operate satisfactorily to obtain a value of m in the order of 0.5 and, hence, can be used in the proposed transmitting end amplifier system. The insertion loss is tolerable and can easily be compensated. The operating range, while not ideal, appears to be better than other simple power law devices.

In designing the receiving end amplifier, there are two methods by which one may perform the inverse operation. The first method consists of using a circuit of the type shown in Figure 14.

E_{in} vs. E_{out} for circuit shown in Figure 6 with R equal to zero ohms

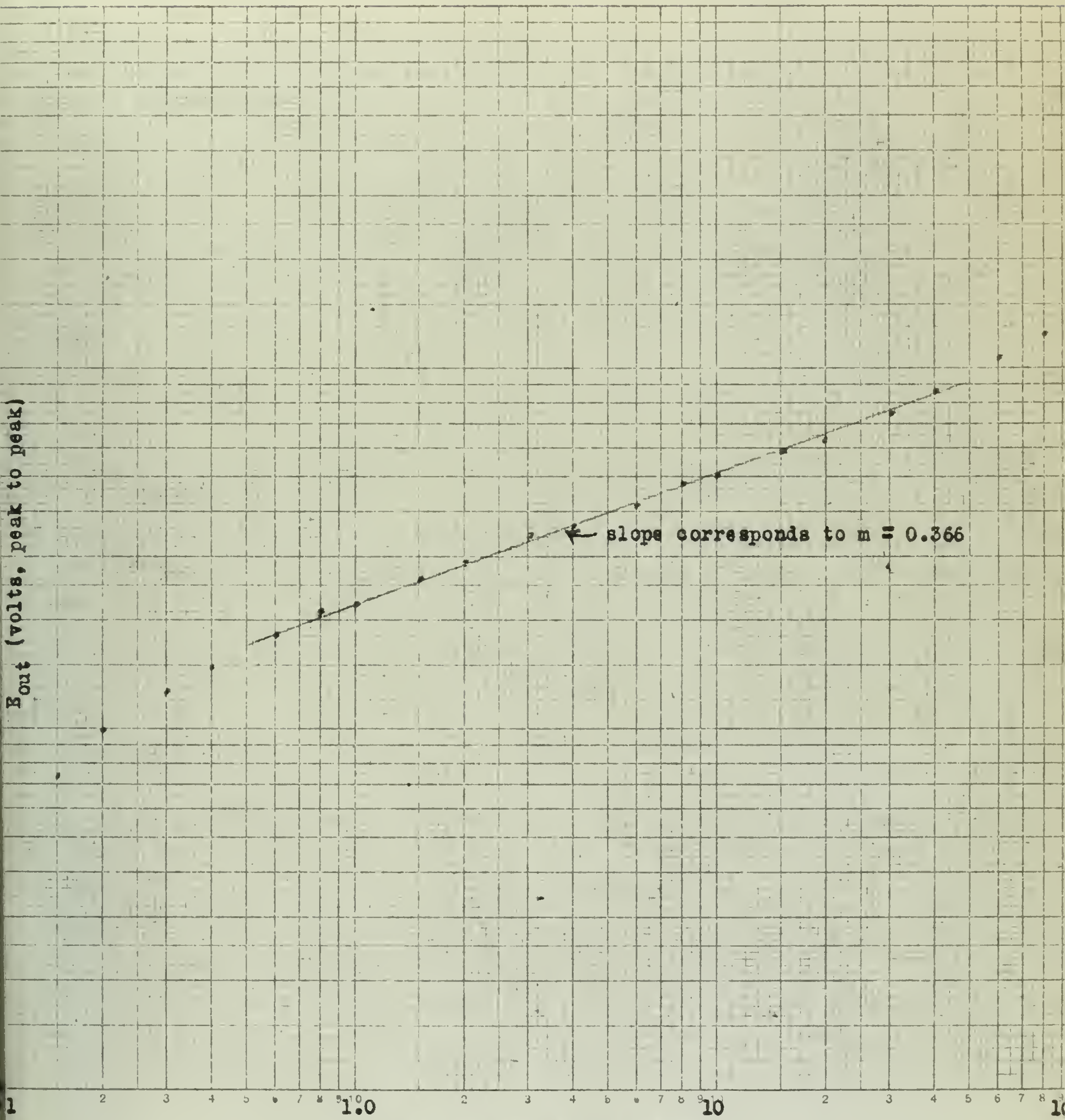


Figure 7

E_{in} vs. E_{out} for circuit shown in Figure 6 with R equal to 100 ohms

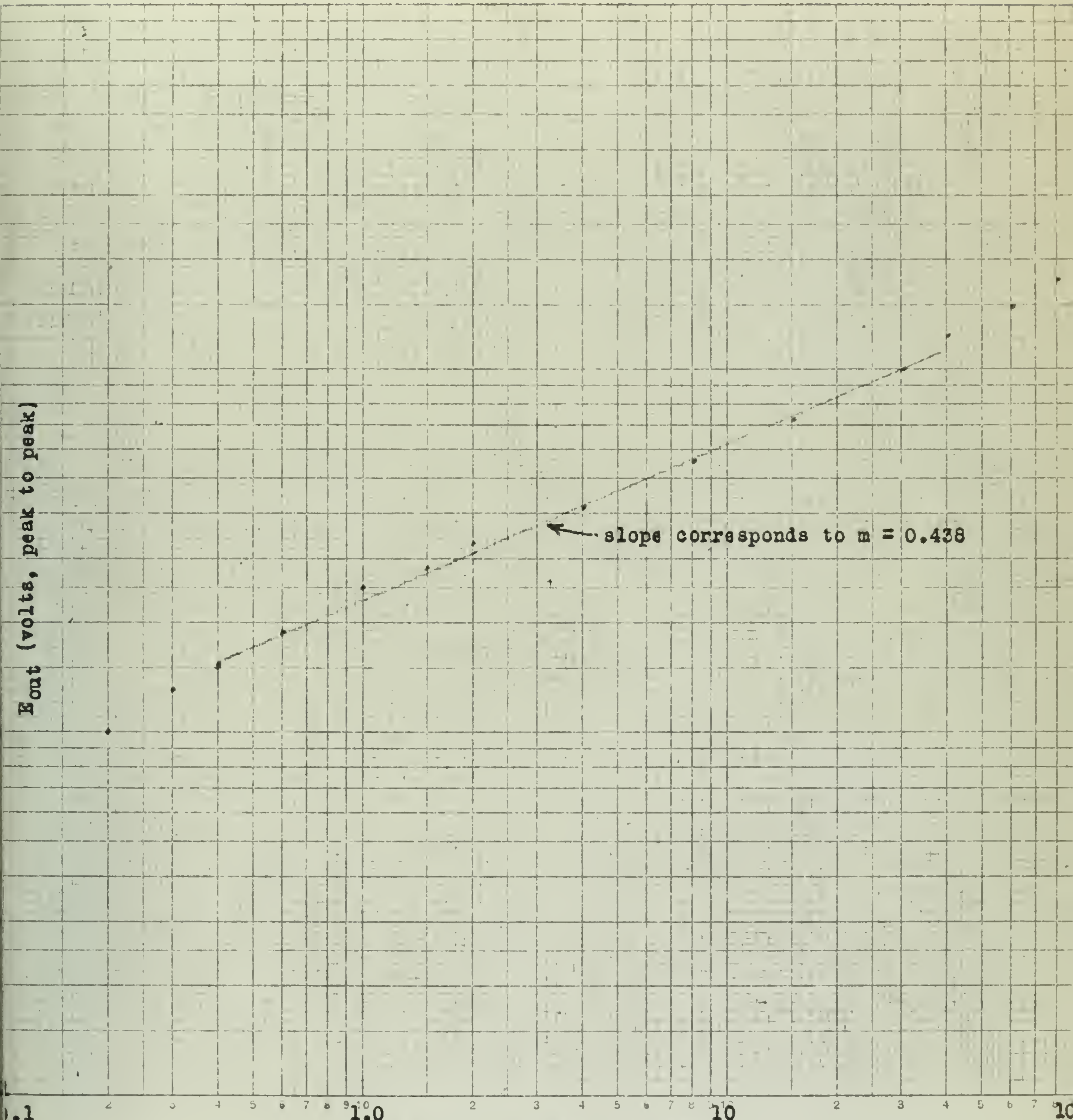


Figure 8

E_{in} vs. E_{out} for circuit shown Figure 6 with R equal to 200 ohms

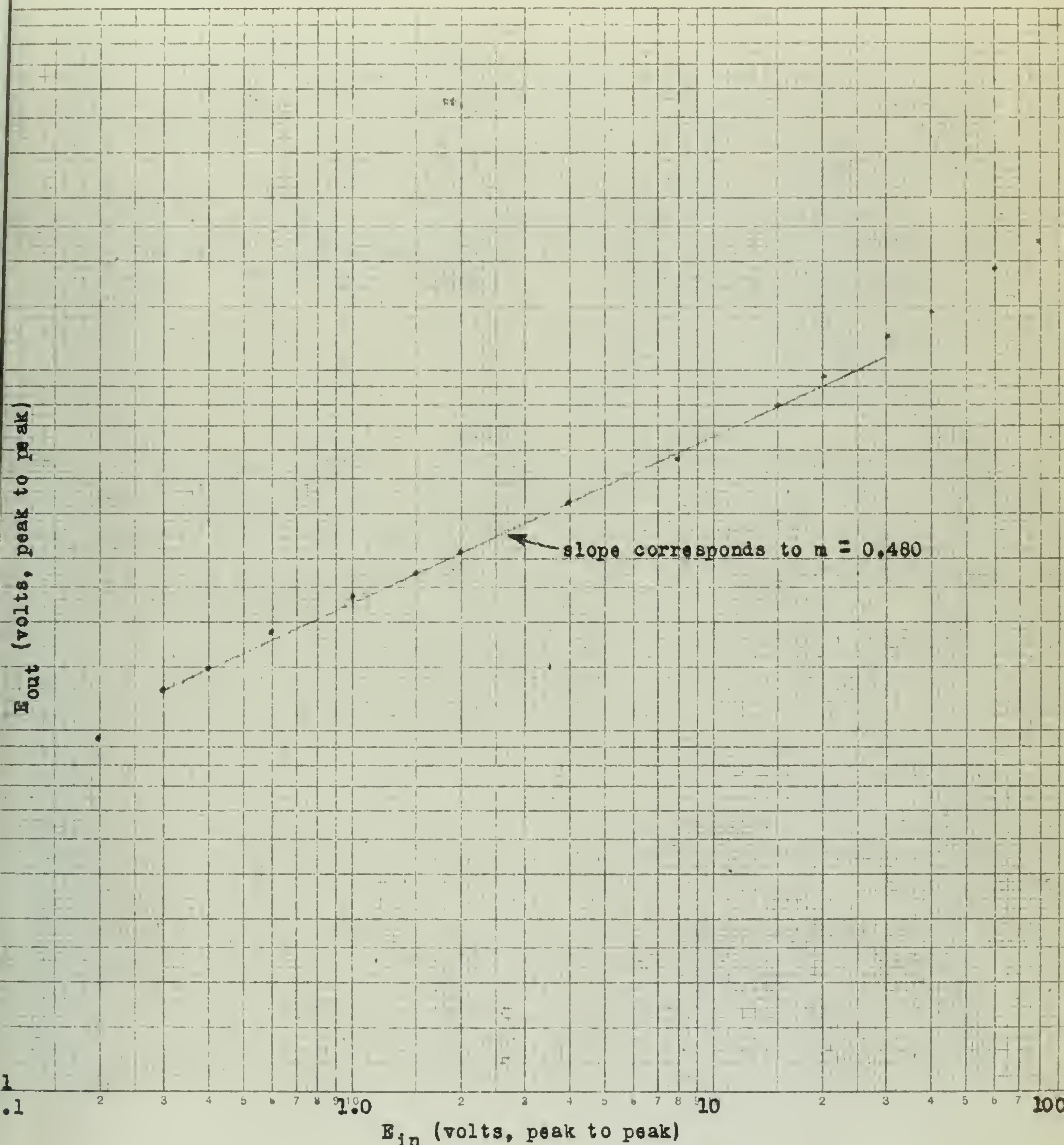


Figure 9

E_{in} vs. E_{out} for circuit shown in Figure 6 with R equal to 300 ohms

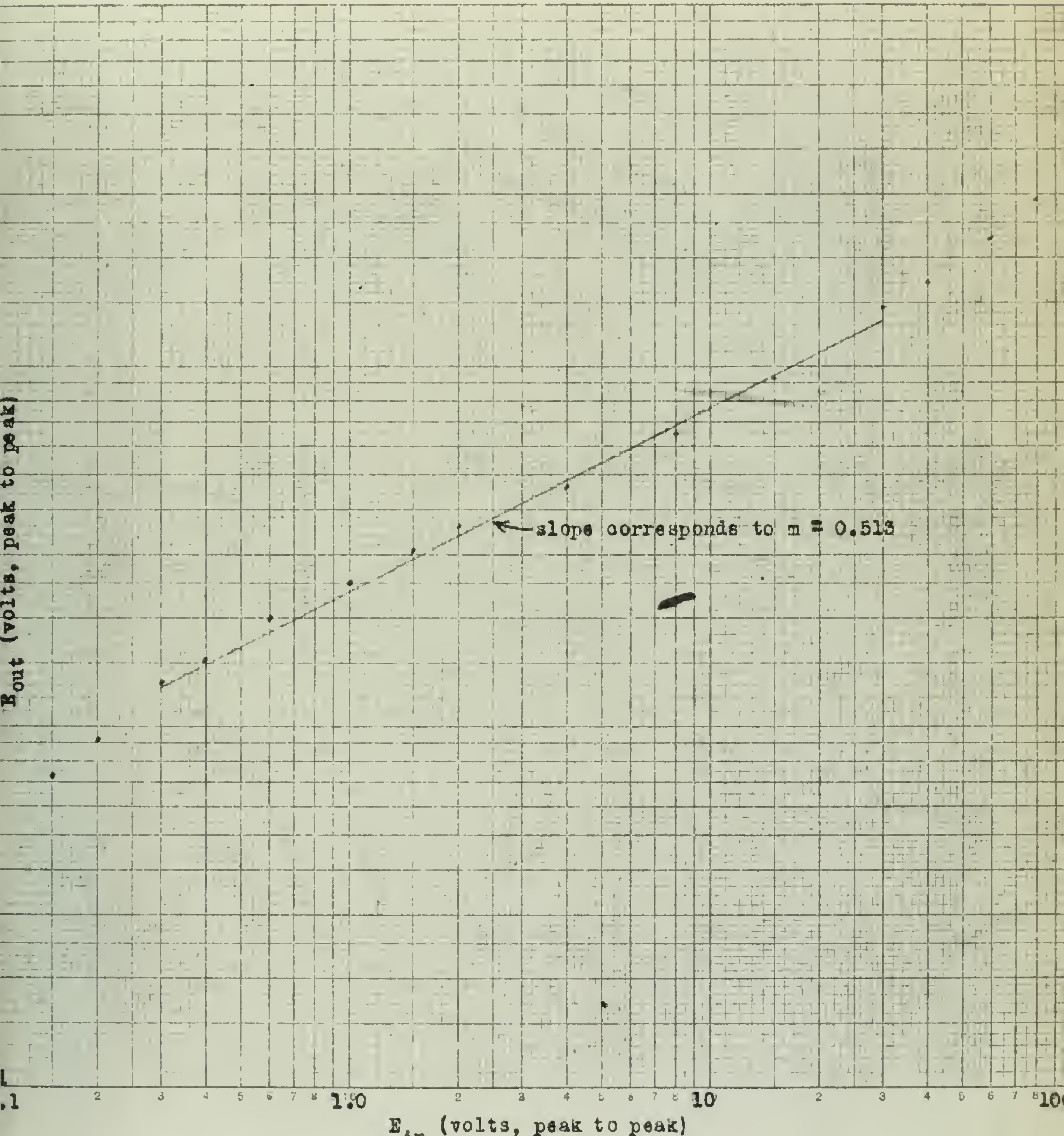


Figure 10

E_{in} vs. E_{out} for circuit shown in Figure 6 with R equal to 400 ohms

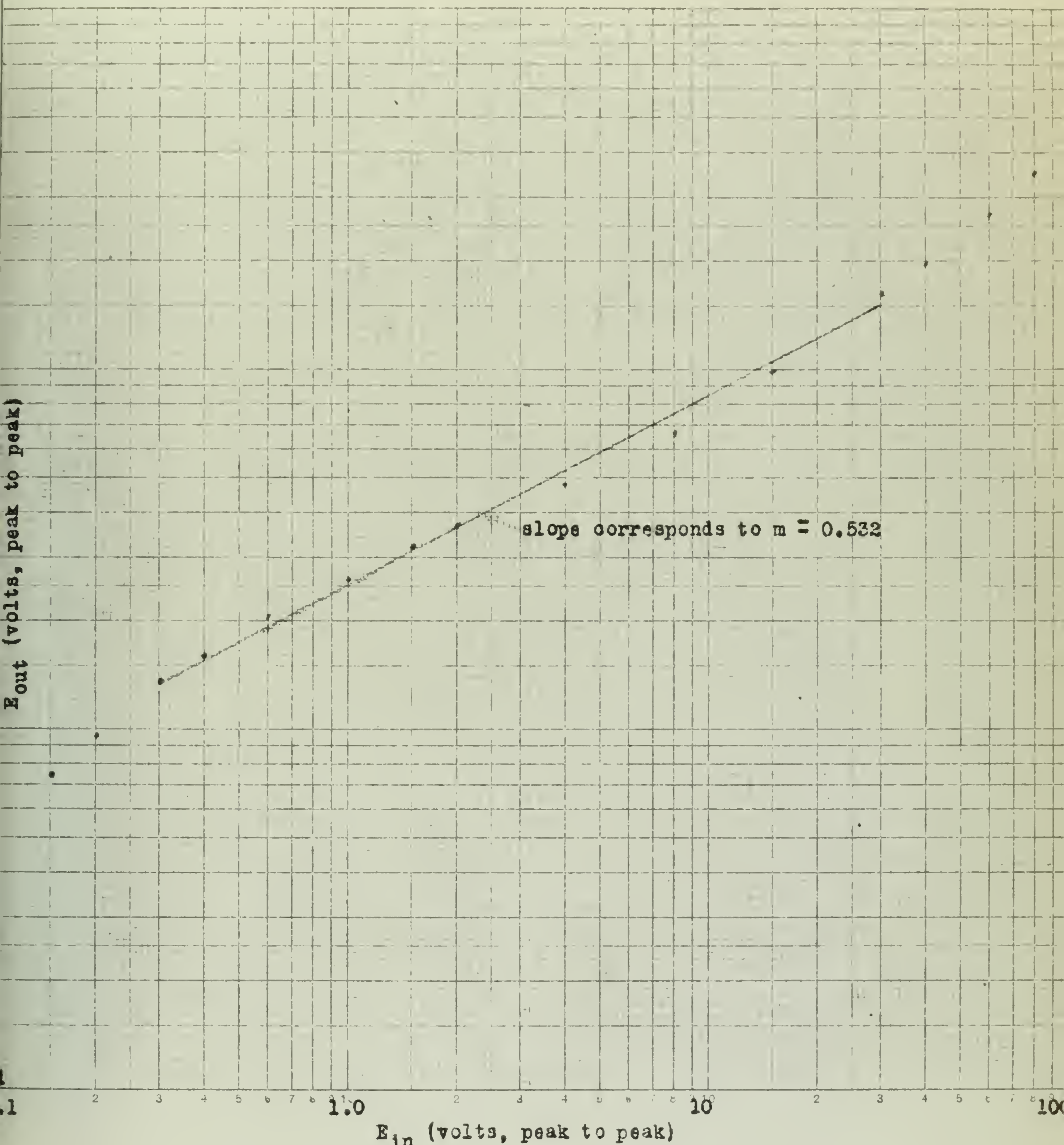
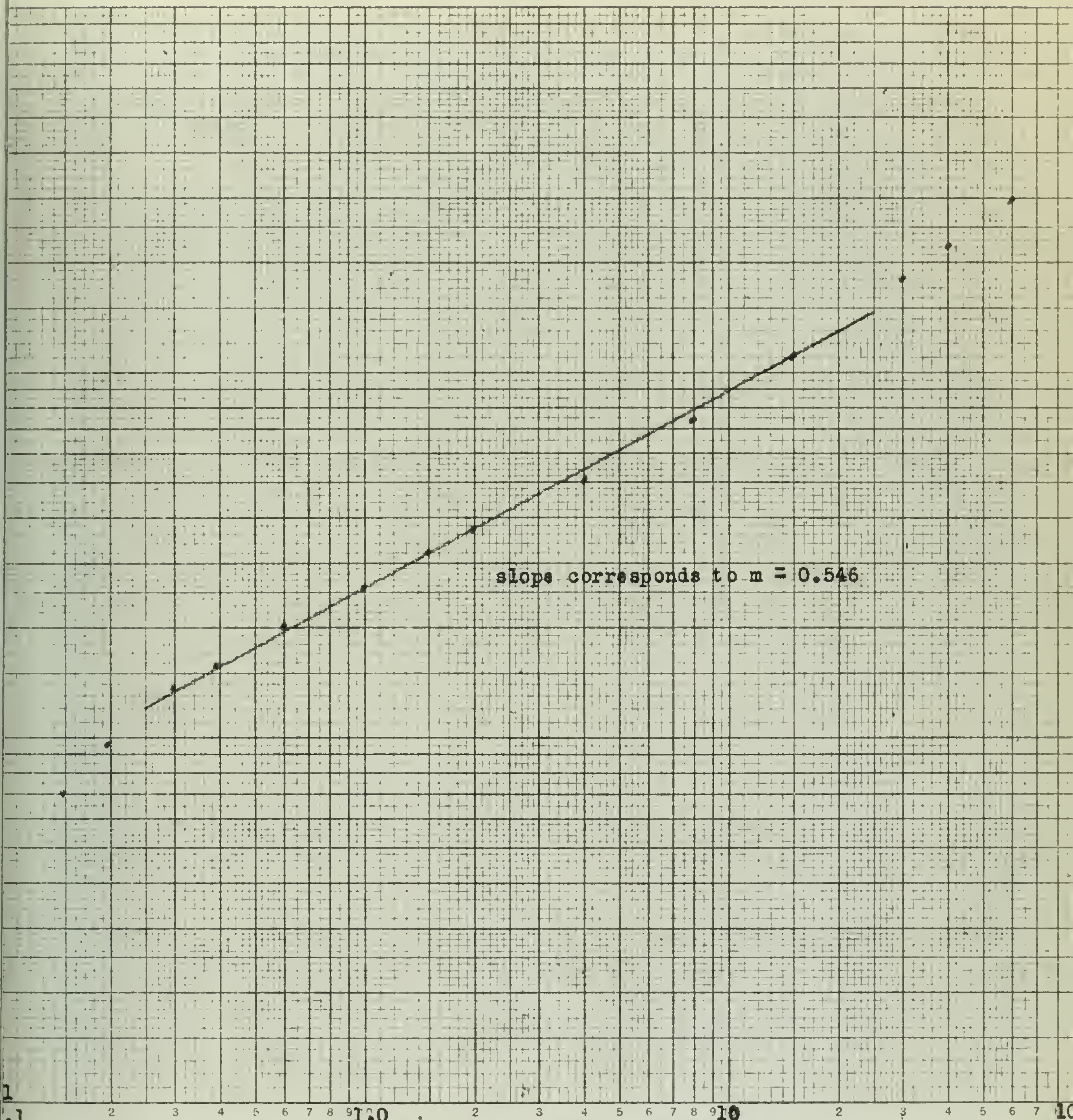


Figure 11

E_{in} vs. E_{out} for circuit shown in Figure 6 with R equal to 500 ohms



E_{in} (volts, peak to peak)

Figure 12

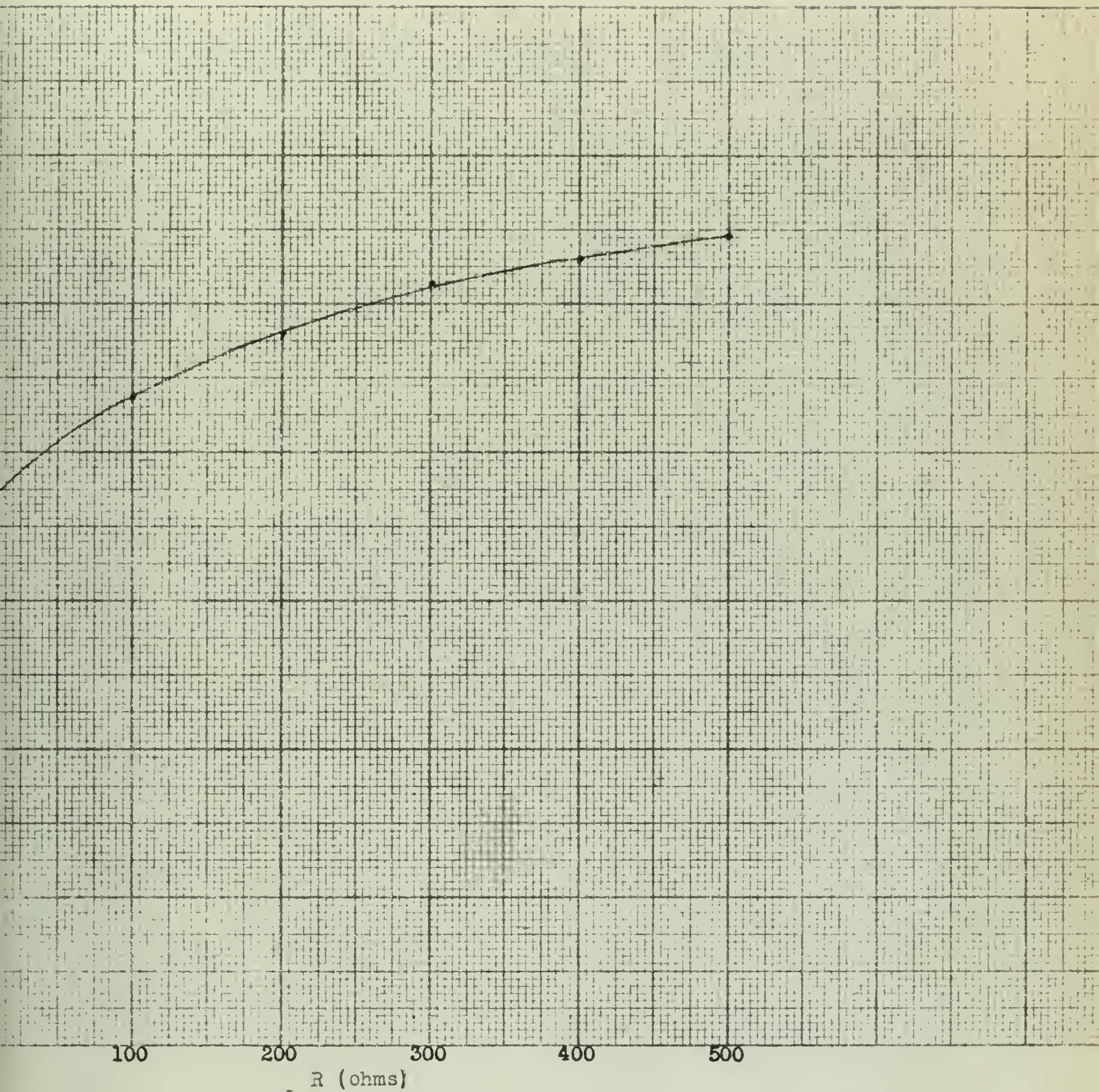


Figure 13.

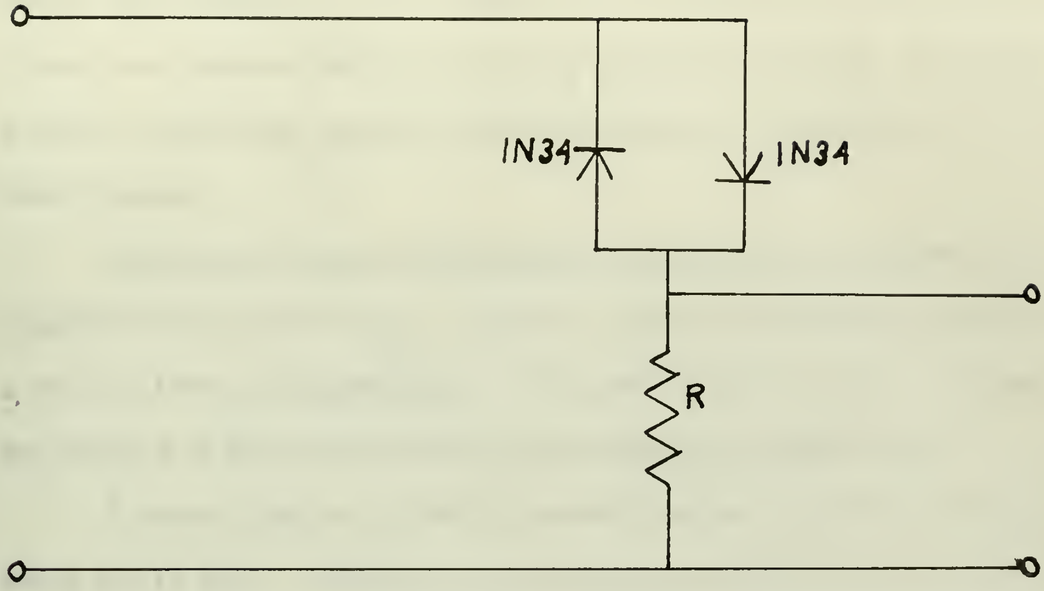


Figure 14.

Circuit for obtaining a response corresponding to Equation 4.

If the resistance R is much less than the resistance of the diodes, then by comparison with Equation 1,

$$I = \left(\frac{E_{in}}{a}\right)^{\frac{1}{m}} \qquad \text{Eq. (3)}$$

However,

$$E_{out} = IR$$

Therefore,

$$E_{out} = R\left(\frac{E_{in}}{a}\right)^{\frac{1}{m}} \qquad \text{Eq. (4)}$$

giving us the desired inverse function.

The circuit shown in Figure 14, however, has the disadvantage of having a rather low input impedance. Furthermore, since it is a non-linear device, this impedance varies. During the peak of the audio

cycle, this impedance will reduce to the order of a few hundred ohms. If one is to prevent this low varying load from causing distortion in the source, the source output impedance must be something less than about thirty ohms.

The value of source impedance can easily be obtained from a transformer output stage. However, before attempting to feed such a circuit from a transformer, due consideration should be given to the effects of phase distortion as discussed in Chapter III.

A bench type set-up of the arrangement was tried and found to work fairly well. However, with sinusoidal input to the overall system, distortion of the output wave became very noticeable below about 400 cycles as the result of the phase distortion within the system.

The second method of performing the inverse operation has a high impedance input and is not as susceptable to phase distortion effects. It consists of using a circuit of the type shown in Figure 6 in a high gain inverse feedback loop. A block diagram of the arrangement is shown in Figure 15.

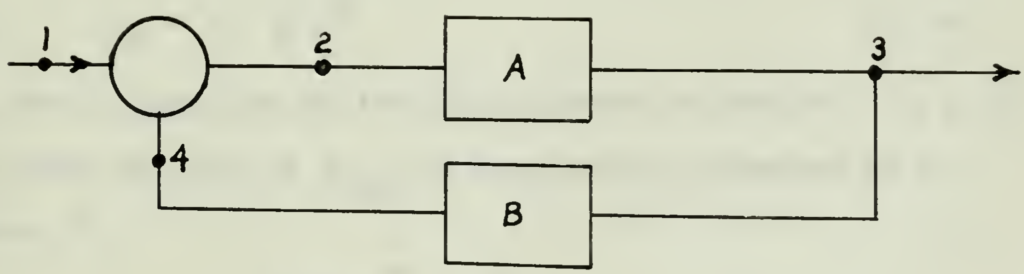


Figure 15.
Block diagram of arrangement for obtaining a response corresponding to Equation 4 utilizing a high gain inverse feedback loop where B contains the circuit of Figure 6, and A indicates the gain between points 2 and 3.

From Figure 15,

$$e_3 = A e_2 \qquad \text{Eq. (5)}$$

$$e_2 = \frac{e_3}{A} \qquad \text{Eq. (6)}$$

By inspection of Figure 15 and Equations (2) and (5),

$$e_4 = a\left(\frac{Ae_2}{R}\right)^m$$

$$e_2 = e_1 - e_4 = e_1 - a\left(\frac{Ae_2}{R}\right)^m$$

$$\frac{e_3}{e_1} \doteq \frac{e_3}{\frac{e_3}{A} + a\left(\frac{e_3}{R}\right)^m} = \frac{A}{1 + Aa(e_3)^{m-1}\left(\frac{1}{R}\right)^m}$$

If $Aa\,(e_3)^{m-1}\left(\frac{1}{R}\right)^m$ is made much greater than unity,

$$\frac{e_3}{e_1} = \frac{A}{Aa(e_3)^{m-1}\left(\frac{1}{R}\right)^m} = \frac{1}{a(e_3)^{m-1}\left(\frac{1}{R}\right)^m} \qquad \text{Eq. (7)}$$

$$e_3 = R\left(\frac{e_1}{a}\right)^{\frac{1}{m}}$$

$$e_3 = R\left(\frac{e_1}{a}\right)^{\frac{1}{m}} \qquad \text{Eq. (8)}$$

This is precisely the function we desire to perform. To prove this, insert the value of E_{out} as determined by Equation (2) in Equation (8)

$$e_3 = R\left(\frac{\frac{a\,(E_{in})^m}{R^m}}{a}\right)^{\frac{1}{m}} = E_{in}$$

Since e_3 is the output voltage of the system shown in Figure 15, this system has performed the exact inverse of the unit shown in Figure 6, giving us an output voltage identical with the voltage at the input of the system.

In order to determine a suitable value of A to validate Equation (7), assume

$$A a \, (e_3)^{m-1} \left(\frac{1}{R}\right)^m \geqq 100 \qquad \text{Eq. (9)}$$

Equation 9 now becomes

$$A(240)(1/R)^m \geqq 100\,(e_3)^{0.5}$$

$$A \geqq \frac{100(e_3)^{0.5}}{(240)(1/R)^m}$$

From Figure 6, $R = 10,000$ ohms. Letting $m = 0.5$ and using Equation (2) and Figure 10

$$a/(10^4)^m = 0.24$$

$$a = 240$$

Figure 10 indicates that the largest value of e_3 will be about 36 volts.

Substituting in Equation (9)

$$A\,(240)(1/6)(1/100) = 100$$

$$A = 2,500$$

A test bench type set-up of this arrangement was made and found to

work satisfactorily. Most of the results indicated in Chapter IV were obtained from this bench type lay out.

The main difficulty with this second method of performing the inverse operation is preventing oscillations. Due to the high gain inverse feedback loop, it is difficult to prevent phase shifts from causing the feedback to become positive feedback at some frequency, and hence cause these oscillations. To prevent phase shifts from accumulating around the feedback loop, one should design the amplifier so that only one stage of the feedback loop is predominant in restricting the pass-band characteristics of the loop. That is, one stage should have only the minimum pass-band characteristics, whereas all other stages should have much wider pass-band characteristics.

A schematic diagram for a complete transmitter amplifier and receiver amplifier arrangement is shown in Figure 16.

Proposed Circuit Schematic for Compressing Amplifier and Expanding Amplifier

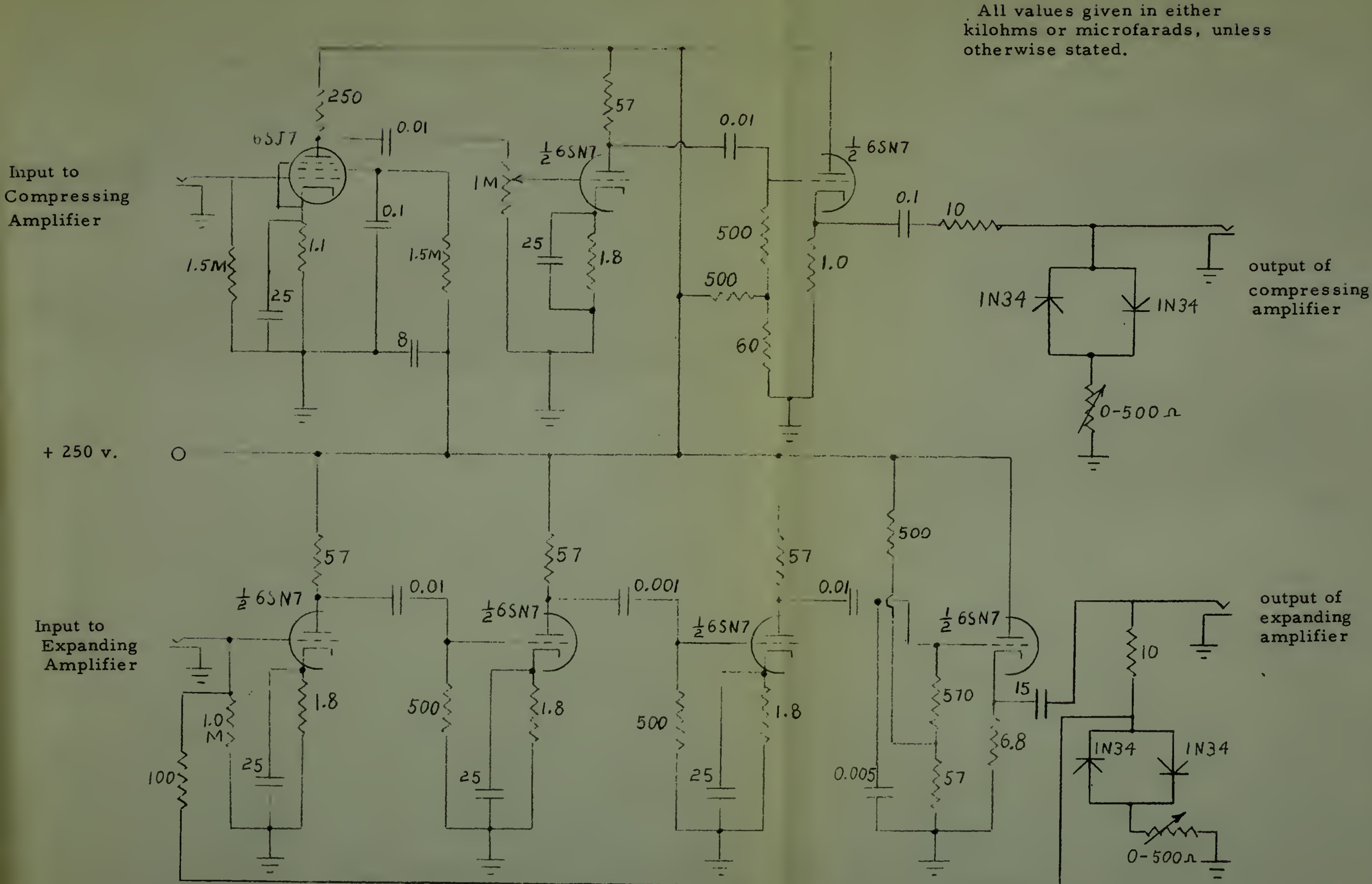


Figure 16

CHAPTER III

MATHEMATICAL ANALYSIS OF CIRCUIT OPERATION

In this chapter, an analysis is made of the operation of the transmitting end amplifier. It is the purpose of this chapter to mathematically determine how distortion of the compressed wave affects the resultant wave after expansion in order that the effects of frequency distortion and phase distortion within the system can be determined. Because of the non-linearities of the system, it is difficult if not impossible to determine any functional relationship for distortion in the output wave caused by distortion between the unit performing the "half-power" function and the unit performing the inverse operation. Consequently, the analysis is made on the basis of a specific amount of distortion occuring within the system.

Although, in general, analysis of any system containing non-linear elements on the basis of any particular wave such as a cosine wave is of problematical value in discussing speech distortion, it is felt that in this case some valuable information can be obtained by utilizing a cosinusoidal input wave. However, a Fourier analysis for the case of a cosinusoidal wave input becomes overly complicated after the first few terms. A Fourier analysis through the fifth harmonic for the compressed cosine wave is shown in Appendix I.

The cosine wave is converted into a wave of the type shown in Figure 17.

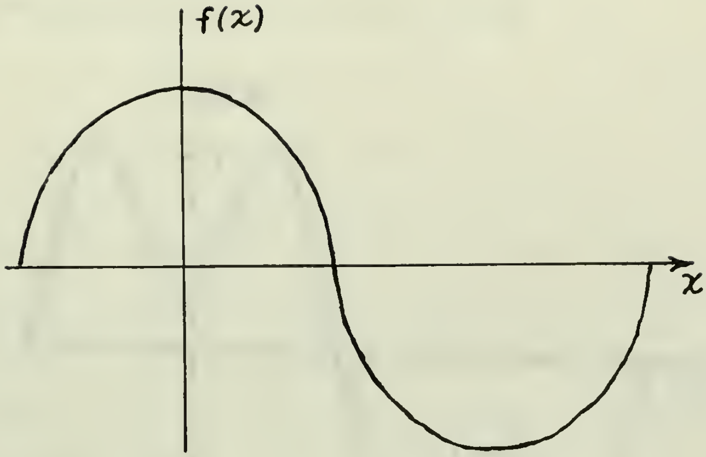


Figure 17

Plot of compressed cosine wave.

The curve of Figure 17 may also be described by the function,

$$f(x) = \left| \frac{\sqrt{|\cos x|}}{\cos x} \right| \cos x$$

As is shown in Appendix I, a wave form of the type shown in Figure 10 has a Fourier Series of the form

$$f(x) = A_1 \cos x + A_3 \cos 3x + A_5 \cos 5x + \ldots + A_n \cos nx$$

Hence, for simplicity, let us assume that we have a wave form of the type

$$f(x) = A \cos x + B \cos 3x \qquad \text{Eq. (1)}$$

appearing at the output of the compressing amplifier. Depending on the magnitude of the third harmonic component, this would have a form similar to the solid curve of Figure 18.

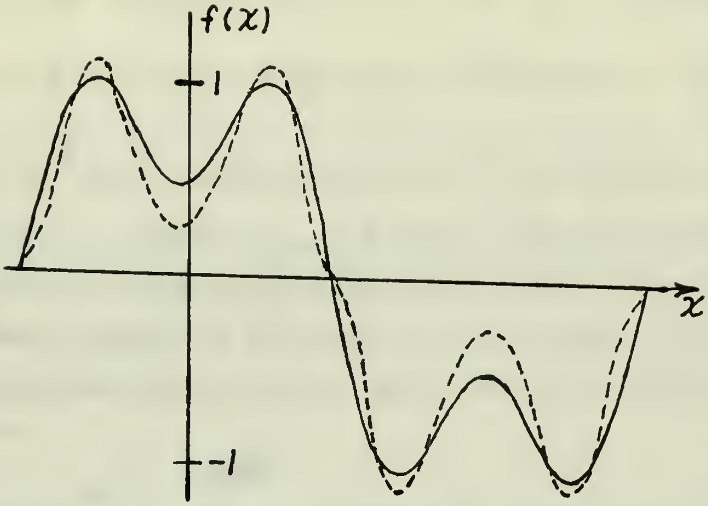


Figure 18

Waveform at output of transmitting end amplifier with input wave form of the type described by Equation 1 of Chapter III; Solid curve represents a typical waveform of the type described by Equation 1. Dashed curve is a plot of the same waveform after passing through the transmitting end amplifier

When this is passed through the amplifier performing the "squaring" function, we obtain a wave form of the type shown by the dashed curve of Figure 18.

One might expect that when a wave form of the type indicated by Equation (1) is passed through the amplifier performing the squaring function, we would obtain a waveform described by

$$\overline{f(x)}^{2} = A^{2}\cos^{2}x + 2AB\cos x\cos 3x + B^{2}\cos^{2}3x \qquad \text{Eq. (2)}$$

However, the amplifier performing the squaring function retains

the sign of the function being squared, i.e., when f(x) becomes negative, $\overline{f_{(x)}}^{2}$ also becomes negative.

Nevertheless, Equation (2) does hold if we assign the power limits, and provided B is less than A. Letting F(x) refer to the output of the amplifier performing the squaring function, then

$$F(x) = +(A^2\cos^2 x + 2AB\cos x\cos 3x + B^2\cos^2 3x); \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad \text{Eq. (3)}$$

$$F(x) = -(A^2\cos^2 x + 2AB\cos x\cos 3x + B^2\cos^2 3x); \quad \frac{\pi}{2} < x < \frac{3\pi}{2} \qquad \text{Eq. (4)}$$

Now, then, by performing a Fourier analysis between these limits, it is possible to determine the nature and extent of the harmonics that should appear at the output of the system. This periodic non-sinusoidal wave may be represented by a Fourier Series as follows:

$$f(x) = \frac{A_o}{2} + \sum_{n=1}^{n=\infty} (A_n \operatorname{Cos} 2n\frac{\pi}{T}x + B_r \sin 2n\frac{\pi}{T}x) \qquad \text{Eq. (5)}$$

where if C is any constant, and P is the period of the wave,

$$A_n = \frac{2}{T}\int_C^{C+P} f(x)\cos\frac{2n\pi x}{P}\,dx \qquad \text{Eq. (6)}$$

$$B_n = \frac{2}{T}\int_C^{C+P} f(x)\sin\frac{2n\pi x}{P}\,dx$$

Taking the region from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $P = 2\pi$, and $C = -\frac{\pi}{2}$

$$f(x) = \frac{A_o}{2} + \sum_{n=1}^{n=\infty} (A_n \cos nx + B_n \sin nx) \qquad \text{Eq. (7)}$$

In the function described by Equations (3) and (4), f(x) = f(-x), and therefore their series consists of cosine terms alone.

Furthermore, f(x) = -f(x + π) and hence its series will contain only odd harmonics.

Substituting the limits in Equation (6),

$$A_n = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{3\pi}{2}} f(x) \cos nx\,dx$$

By the same method used in developing Equation (4) of Appendix I, it can be shown that

$$A_n = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \cos nx dx \qquad \text{Eq. (8)}$$

Referring to Equation (3), let

$$g = A^2 \cos^2 x \qquad \text{Eq. (9)}$$

$$h = 2AB \cos x \cos 3x \qquad \text{Eq. (10)}$$

$$i = B^2 \cos^2 3x \qquad \text{Eq. (11)}$$

so that

$$F(x) = g + h + i \qquad \text{Eq. (12)}$$

Let A_n (g) refer to the Fourier coefficients of g.

$$A_n(g) = \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} A^2 \cos^2 x \cos nx dx \qquad \text{Eq. (13)}$$

$$A_1(g) = \frac{2A^2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \mathrm{Cos}^3\, x dx \qquad a \qquad \text{Eq. (14)}$$

$$= \frac{2A^2}{4\pi} \left[1/3 \sin 3x + 3 \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$A_1(g) = 2\frac{4A^2}{3\pi} \qquad \text{Eq. (15)}$$

$$A_3(g) = \frac{2A^2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \cos 3x \, dx$$

$$A_3(g) = \frac{2A^2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (4\cos^5 x - 3\cos^3 x)\, dx \qquad \text{Eq. (16)}$$

By trigonometric substitution and Equations (14) and (15),

$$A_3(g) = \frac{2A^2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 4(1 - 2\sin^2 x \quad \sin^4 x) \cos x dx - \frac{8A^2}{\pi}$$

$$= \frac{2A^2}{\pi}\left[(4\sin x - 8/3\sin^3 x + 4/5\sin^5 x \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \frac{8A^2}{\pi}$$

$$A_3(g) = 2\,\frac{4A^2}{15\pi} \qquad \text{Eq. (17)}$$

$$A_5(g) = \frac{2A^2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \cos 5x \, dx$$

$$A_5(g) = \frac{2A^2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (16\cos x - 20\cos^5 x + 5\cos^3 x)\, dx \qquad \text{Eq. (18)}$$

By Equations (14), (15), (16) and (17),

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx = \frac{16}{15} \qquad \text{Eq. (19)}$$

By trigonometric substitution and Equations (14), (15) and (19),

$$A_5(g) = \frac{2A^2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 16(1-3\sin^2 x + 3\sin^4 x - \sin^6 x)\cos x dx$$

$$+ 2\left(\frac{-64}{3\pi} A^2 + \frac{20}{3\pi} A^2\right)$$

$$A_5(g) = \frac{2A^2}{\pi}\left[16 \sin x - 16 \sin^3 x + \frac{48}{5}\sin^5 x - \frac{16}{7} \sin\ x\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$- \frac{88}{3\pi} A^2$$

$$A_5(g) = - 2\ \frac{4A^2}{105\pi} \qquad \text{Eq. (20)}$$

Since this Fourier Series is obviously converging very rapidly, no terms beyond the $A_5(g)$ term will be computed.

Let $A_n(h)$ refer to the Fourier components of h.

Then, by Equations (8) and (10),

$$A_n(h) = \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2\,AB \cos x \cos 3x \cos nx dx \qquad \text{Eq. (21)}$$

$$A_1(h) = \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2AB \cos^2 x \cos 3x dx \qquad \text{Eq. (22)}$$

$$= \frac{4AB}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (4\cos^5 x - 3\cos^3 x)\, dx$$

By Equations (14) and (17),

$$A_1(h) = \frac{4AB}{\pi}\left(\frac{4x16}{15} - \frac{3x4}{3}\right)$$

$$A_1(h) = 2\ \frac{8}{15\pi} \qquad \text{Eq. (23)}$$

By Equation (21)

$$A_3(h) = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2\ AB\ \cos x \cos^2 3x\ dx$$

$$= \frac{4AB}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x\left(\frac{1}{2} + \frac{1}{2}\cos 6\ x\right) dx$$

$$A_3(h) = \frac{4AB}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left[\ \frac{1}{2}\cos x + \frac{1}{2}(32 \cos^1 x - 48 \cos^5 x \right. \qquad \text{Eq. (24)}$$

$$\left. + 18 \cos^3 x - \cos x)\ \right] dx$$

By Equations (18), (14), (15), (19) and (20):

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \text{Cos}\quad x\ dx = \frac{1}{16}\left(\frac{-4}{105} + \frac{20x16}{15} - \frac{5x4}{3}\right.$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\quad x\ dx = \frac{32}{35} \qquad \text{Eq. (25)}$$

By Equations(14), (15), (19), (23) and (24):

$$A_3\ (h) = \frac{4AB}{\pi}\left[\ \frac{16x32}{35} - \frac{24x16}{15} + \frac{9x4}{3}\ \right]$$

$$A_3(h) = 2\ \frac{2AB(36)}{35\pi} \qquad \text{Eq. (26)}$$

By Equation (13),

$$A_5(h) = \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2\ AB \cos x \cos 3x \cos 5x\, dx$$

$$A_5(h) = \frac{4AB}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (64 \cos^9 x - 128 \cos\ x + 80 \cos^5 x - 15 \cos^3 x)\, dx \qquad \text{Eq. (27)}$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^9 x\, dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1 - 4 \sin^2 x + 6 \sin^4 x - 4 \sin^6 x \sin^8 x) \cos x\, dx$$

$$= \left[\sin x - \frac{4}{3} \sin^3 x + \frac{6}{5} \sin^5 x - \frac{4}{7} \sin^7 x + \frac{1}{9} \sin^9 x \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^9 x\, dx = \frac{256}{315} \qquad \text{Eq. (28)}$$

By Equations (14), (15), (19), (25), (27) and (28):

$$A_5(h) = 2\ \frac{2AB}{\pi} \left(\frac{64x256}{315} - \frac{128x32}{35} + \frac{80x16}{15} - \frac{15x4}{3} \right)$$

$$A_5\ (h) = 2\ \frac{2AB}{\pi} \left(\frac{20}{63} \right) \qquad \text{Eq. (29)}$$

By Equations (8) and (11)

$$A_n\ (i) = 2/\pi \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} B^2 \cos^2 3x \cos nx\, dx$$

$$A_1(i) = B\ ^2/_\pi \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x + \cos 6x) \cos x\, dx$$

$$= B^2/\pi \left[\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx + \frac{1}{2} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos 5x\, dx + \frac{1}{2} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos 7x\, dx \right] = 2\,B^2/\pi\,(1 + 1/10 - 1/14)$$

$$= 2\,B^2/\pi\,(36/35)$$

Similarly

$$A_3(i) = 2B^2/\pi\,(-1/3 - 1/6 + 1/18)$$

$$= -2\,B^2/\pi\,(4/9)$$

$$A_5(i) = 2\,B^2/\pi\,(1/5 + \tfrac{1}{2} - 1/22)$$

$$A_5(i) = 2\,B^2/\pi\,(36/55) \qquad \text{Eq. (29)}$$

Although this series does not seem to be converging, computations indicate that $A_7(i)$ equals approximately $2B^2/\pi\,(5/14)$ and $A_9(i)$ equals $2\,B^2/\pi\,(-4/45)$ and thereafter the series converges very rapidly. The $A_7(i)$ and $A_9(i)$ terms will not be used, however, as the 7th and 9th order terms for g and h were determined. Furthermore, in the usual case, B will be of small magnitude which when squared and multiplied by the coefficients $2/\pi(5/14)$ and $2/\pi\,(4/45)$ respectively, makes these terms insignificantly

small. For the case of a sinusoidal input, Appendix I shows B equals to (-0.160).

By Equations (3), (5) and (13),

$$F(x) = \sum_{n=1}^{\infty} [A_n(g) \cos nx + A_n(h) \cos nx + A_n(i) \cos nx]$$

$$= 2 \Big[\frac{4A^2 \cos x}{3\pi} + \frac{4A^2 \cos 3x}{15\pi} - \frac{4A^2 \cos 5x}{105\pi} + \frac{8AB \cos x}{15\,\pi}$$

$$- \frac{2AB(36)\cos 3x}{35\,\pi} - \frac{2AB(20)\cos 5x}{63\,\pi} - \frac{B^2(36)\text{cox } x}{35\,\pi}$$

$$- \frac{B^2(4)\cos 3x}{9\pi} \quad \frac{B^2(36)\cos 5x}{55\,\pi} \Big] \qquad \text{Eq. (30)}$$

$$F(x) = 2(4A^2/3\,\pi + 8AB/15\pi + 36B^2/35\pi) \cos x$$

$$+ 2(4A^2/15\,\pi + 2AB(36)/35\pi - 4B^2/9\pi) \cos 3x$$

$$+ 2(-4A^2/105\pi + 2AB(20)/63\pi - 36B^2/35\pi) \cos 5x \qquad \text{Eq. (31)}$$

This completes the Fourier Series up through terms of the fifth order for the waveform indicated by the dashed line in Figure 18.

If we substitute the values for A and B as determined in Appendix I for the compressed cosine wave, then $A = A_1 = 1.12$ and $B = B_1 = 0.160$. Let $F_c(x)$ refer to the case of a cosinusoidal wave at the input of this system, and equation (31) becomes

$$F_c(x) = 2\left[\frac{4(1.12)^2}{2\pi} - \frac{8(1.12)(0.160)}{15\pi} + \frac{36(0.160)^2}{35\pi}\right]\cos x$$

$$2\left[\frac{4(1.12)^2}{15\pi} - \frac{2(1.12)(0.160)(36)}{35} - \frac{4(0.160)^2}{9}\right]\cos 3x$$

$$2\left[\frac{-2(1.12)(0.160)(20)}{63\pi} - \frac{4(1.12)^2}{105\pi} + \frac{36(0.160)^2}{55\pi}\right]\cos 5x$$

Eq. (32)

We have here now a Fourier Series up through terms of the fifth order for the wave at the output of the receiving end amplifier with a sine wave input to the transmitting end amplifier. If Equation (32) included all significant harmonics, it should therefore be a sinusoidal wave, or it should simply equal cos x. In order for this to be true, the coefficient of cos x in Equation 32 must converge to unity and the coefficients of all other terms must converge to zero. It is interesting to note how nearly true this is in Equation (32). It should be noted that since Equation (3) and (4) did not include any harmonics above the third harmonic, Equation (32) is representative of the case where an idealized filter having no phase shift and a cut-off frequency just above the third harmonic is inserted between the sending end amplifier and the receiving end amplifier.

Computing the coefficients of Equation (32) we obtain:

$$F_c(x) = 1.021 \cos x - 0.028 \cos 3x - 0.0922 \cos 5x. \qquad \text{Eq. (33)}$$

This shows a surprisingly close reproduction of the cos x wave. Note, that whereas the third harmonic distortion is only about 2.8% the fifth harmonic distortion is about 9%. Experimental evidence seems to indicate that this relatively large fifth harmonic is indicative of how the system generally operates and hence an explanation of it is in order.

If the system operates perfectly so that it does not introduce any distortion of the input wave other than that caused by the functions it is intended to perform, then all distortion introduced by the sending end amplifier is removed by the receiving end amplifier. From Appendix I, it is apparent that if we have a sinusoidal input wave to this "perfect" system, there will exist at the input to the receiving end amplifier, a waveform that may be described by

$$f(x) = A_1 \cos x + A_3 \cos 3x + A_5 \cos 5x + \ldots + A_n \cos nx + \ldots$$

where the limits of Equations 3 and 4 apply.

At the output of such a system, there will be a waveform of the type

$$F_c(x) = + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \cos nx \cos mx, \quad \begin{matrix} n = 1, 3, 5, .. \\ m = 1, 3, 5, .. \\ \frac{-\pi}{2} < x < \frac{\pi}{2} \end{matrix}$$

$$n = 1, 3, 5, \ldots, \quad m = 1, 3, 5, \ldots$$

$$F_c(x) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \cos nx \cos mx, \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Each term of the equations for $F_c(x)$ can be described by a Fourier Series. Letting $F_{c_{nm}}(x)$ refer to a particular term of the equations for $F_c(x)$, we may say

$$F_{c_{nm}}(x) = A_n A_m (a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \ldots + a_p \cos px + \ldots)$$

Referring to any specific harmonic as the kth harmonic, we see that for every pair of values of m and n there is a finite mathematical contribution to the kth harmonic having a magnitude

$A_nA_ma_k$. However, since in the "perfect" system, the kth harmonic will not exist in the output, the sum of all such terms must be equal to zero, that is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_nA_ma_k = 0.$$

If we refer to the highest harmonic that the system is capable of passing as the Lth harmonic, then neither m nor n can exceed L. By comparison with the Fourier coefficients that have been computed, it is apparent that most of the significant coefficients contributing to the kth harmonic arise from those terms for which either n or m or both are less than or equal to k. Therefore, if L is greater than k, then most of the kth order terms having significant coefficients will appear "mathematically" in the output, and we can expect that their sum will be near zero. That is, there will be almost perfect cancellation of the kth order harmonic. If, however, L is somewhat less than k a smaller portion of the kth order terms having significant coefficients will appear in the output, and the cancellation will not be as perfect, and the magnitude of the kth order harmonic may exceed the magnitudes of preceding lower order harmonics. For harmonics of higher order then k , however, the magnitude of the harmonics appearing in the output decreases very rapidly despite the imperfect cancellation because the coefficients $A_nA_ma_p$ become very small.

Consequently, with a sinusoidal input wave, an idealized low

pass filter having no phase shift and a cut-off frequency just below the kth order harmonic can be expected to cause the harmonic content of the output wave to be mainly above the kth order harmonic. Therefore, a similar filter in the output of the system will eliminate most of the distortion caused by the first filter.

If this were a linear system, we could then apply the laws of superposition and expect a similar result from a complex wave such as a speech wave. We could then assume that if we restricted the transmitted wave bandwidth by inserting such an ideal low pass filter having a cut-off frequency of three kilocycles, the main harmonics appearing in the output wave would be above three kilocycles. We could then eliminate these unwanted harmonics by simply passing the output wave through another low pass filter having a three kilocycle cut-off frequency.

Since the transmitting end amplifier and the receiving end amplifier are both non-linear in nature, a complex wave entering either amplifier will result in cross modulation components, consisting of sum and difference frequencies. If there is no other distortion within the system, however, these cross modulation components must all be in such phase and amplitude at the output as to be cancelled. If a filter is inserted after the transmitting end amplifier, some of the cross modulation frequencies will not pass through the system, and consequently there will be imperfect cancellation of these frequencies in the output of the system. In general there will be cross modulation components generated by

the transmitting end amplifier that will lie within the pass band of any band width limiting filter. It can be expected that these components will give rise to imperfectly cancelled components of the same frequency in the output. Therefore, these frequencies will not be eliminated by a similar filter in the output circuit and distortion will result. Nevertheless, experimental results reported in Chapter IV seem to indicate that the system operates very similar to a linear system in this respect and consequently a similar filter in the output circuit does improve the quality of reproduction with limited bandwidth. That is, the system seems to operate very nearly as if the laws of superposition did apply. Since it is impossible at this state of the art to build a filter that does not have a phase shift, a rigorous experimental verification of this is hard to obtain.

So far, we have considered using only an ideal filter. Now let us assume that we have a filter, or imperfect audio system, that introduces no phase distortion but does reduce the amplitude of the third harmonic in equation 1 by 10%, and reduces all higher order harmonics to zero. Hence,

$$f(x) = A \cos x + 0.9 B \cos 3x \qquad \text{Eq. (34)}$$

The equations corresponding to Equations 3 and 4 become

$$\overline{f(x)}^{2} = (A^2 \cos^2 x + 1.8 AB \cos x \cos 3x + 0.81 B^2 \cos^2 3x)$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \qquad \text{Eq. (35)}$$

$$\overline{f(x)}^2 = -(A^2 \cos^2 x + 1.8\, AB \cos x \cos 3x + 0.81\, B^2 \cos^2 3x)$$

$$\frac{\pi}{2} < x < \frac{3}{2}\pi \qquad \text{Eq. (36)}$$

To obtain equations equivalent to Equations (9), (10) and (11), let

$$g' = A^2 \cos^2 x \qquad \text{Eq. (37)}$$

$$h' = 1.8\, AB \cos x \cos 3x \qquad \text{Eq. (38)}$$

$$i' = 0.81\, B^2 \cos^2 3x \qquad \text{Eq. (39)}$$

Letting $A_n(g')$, $A_n(h')$ and $A_n(i')$ refer to the respective Fourier coefficients of g', h' and i', it is apparent that

$$A_n(g') = A_n(g) \qquad \text{Eq. (40)}$$

$$A_n(h') = 0.9\, A_n(h) \qquad \text{Eq. (41)}$$

$$A_n(i') = 0.81\, A_n(i) \qquad \text{Eq. (42)}$$

Let $F_c(x')$ refer to the case of a cosinusoidal wave at the input of this system when this non-ideal filter is inserted in the system between the transmitting end amplifier and receiving end amplifier. Then by Equations 31, 40, 41 and 42, an equation equivalent to Equation (32) is

$$F_c(x') = 2\left[\frac{4(1.12)^2}{3\pi} - \frac{8(0.9)(1.12)(0.160)}{15\pi} + \frac{(0.81)(36)(0.160)^2}{35\pi}\right]\cos x$$

$$+ 2\left[\frac{4(1.12)^2}{15\pi} - \frac{2(0.9)(1.12)(0.160)(36)}{35\pi} - \frac{4(0.81)(0.160)^2}{9\pi}\right]\cos 3x$$

$$2\left[-\frac{2(0.9)(1.12)(0.160)(20)}{63\pi} - \frac{4(1.12)^2}{105\pi} + \frac{(0.81)(36)(0.160)^2}{55\pi}\right]\cos 5x$$

$$\text{Eq. (43)}$$

The equation corresponding to Equation (33) is

$$F_c(x') = 1.025 \cos x - 0.0031 \cos 3x - 0.0868 \cos 5x \quad \text{Eq. (44)}$$

We see that the third harmonic now has an amplitude equal to 0.3% of the fundamental component compared to 2.8% of the fundamental component in Equation (33). The fifth harmonic now has an amplitude equal to 8.32% of the fundamental component compared to 9.0% of the fundamental component in Equation (33). Hence, we see that there has been a slight reduction in the harmonic components. If as Chapter IV seems to indicate, these results are also applicable to the case of speech input, we can conclude that a small amount of amplitude distortion between the compressing amplifier and the expanding amplifier does not appreciably alter the quality of the output of the system.

Now let us consider the case where we have a filter or imperfect audio system that introduces phase distortion along with a small amount of amplitude distortion. This, of course, is the usual case that would be encountered in practice. The amount of phase distortion that would accompany the 10% reduction in the third harmonic component as given in Equation (44) might well be in the order of 30°. Hence, in this case,

$$\begin{aligned} f(x) &= A \cos x + 0.9\, B \cos(3x + 30^\circ) \\ &= A \cos x + 0.9\, B(\cos 3x \cos 30^\circ - \sin 3x \sin 30^\circ) \\ &= A \cos x + 0.787\, B \cos 3x - 0.45\, B \sin 3x \end{aligned}$$

$$f(x) = A \cos x + B\,(0.787 \cos 3x - 0.45 \sin 3x) \quad \text{Eq. (45)}$$

The equations corresponding to Equations (11) and (12) now become

$$\overline{f(x)}^2 = (A \cos x)^2 + 2\ AB \cos x(0.787 \cos 3x - 0.45 \sin 3x)$$
$$+ B^2 (0.787 \cos 3x - 0.45 \sin 3x)^2 \qquad \text{Eq. (46)}$$

$$\underline{f(x)}^2 = - (A \cos x)^2 - 2\ AB \cos x\ (0.787 \cos 3x - 0.45 \sin 3x)$$
$$- B^2 (0.787 \cos 3x - 0.45 \sin 3x)^2 \qquad \text{Eq. (47)}$$

To obtain equations equivalent to Equations (9), (10), and (11), let

$$g'' = A^2 \cos^2 x \qquad \text{Eq. (48)}$$

$$h'' = 2\ AB \cos x\ (0.787 \cos 3x - 0.45 \sin 3x) \qquad \text{Eq. (49)}$$

$$i'' = B^2 (0.787 \cos 3x - 0.45 \sin 3x)^2$$

Letting $A_n(g'')$, $A_n(h'')$ and $A_n(i'')$ refer to the respective Fourier coefficients of g'', h'', and i'', it is apparent that

$$A_n(g'') = A_n(g) \qquad \text{Eq. (51)}$$

From Equations (10), (13) and (50),

$$A_n(h'') = 0.787\ A_n(h) - \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.9\ AB \cos x \cos nx \cos 3x\, dx$$

Let

$$a_n(h'') = \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.9\ AB \cos x \cos nx \sin 3x\, dx$$

$$a_1(h'') = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 0.9\ AB \cos^2 x \sin 3x\, dx$$

$$= \frac{0.9AB}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (3 \cos^2 x \sin x - 4 \cos^2 x \sin^3 x)\, dx$$

$$= \frac{0.9\ AB}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (3 \cos^2 x \sin x - 4 \cos^2 x \sin x + 4 \cos^4 x \sin x)\, dx$$

$$= 0$$

Since all terms in the integral for $a_n(h)$ will reduce to the form,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^n x \sin x\, dx\,, \qquad a_n(h) = 0$$

Hence

$$A_n(h'') = 0.787\ A_n(h) \qquad \text{Eq. (52)}$$

From Equation (50),

$$i'' = B^2(0.611 \cos^2 3x - 0.707 \cos 3x \sin 3x + 0.202 \sin^2 3x$$

$$= (0.611 \cos^2 3x + 0.202 - 0.202 \cos^2 3x - 0.707 \cos 3x \sin 3x$$

$$i'' = B^2(0.202 + 0.409 \cos^2 3x - \frac{0.699}{2} \sin 6x \qquad \text{Eq. (53)}$$

Let

$$a(i'') \pi\ 0.202\ B^2 \qquad \text{Eq. (54)}$$

$$b(i'') = B^2 0.409 \cos^2 3x \qquad \text{Eq. (55)}$$

$$c(i'') = B^2\ 0.349 \sin 6x \qquad \text{Eq. (56)}$$

Remembering that Equations (54), (55) and (56) are restricted by the conditions of Equations (46) and (47), let $a_n(i'')$, $b_n(i'')$ and $c_n(i'')$ refer to the respective Fourier coefficients of $a(i'')$, $b(i'')$ and $c(i'')$ respectively. Then,

$$A_n(i'') = a_n(i'') + b_n(i'') + c_n(i'') \qquad \text{Eq. (57)}$$

Due to the conditions imposed upon Equation (54), $a(i'')$ is actually a square wave having a period of 2 and an amplitude of $0.202B^2$. From the Fourier analysis of a square wave,

$$a(i'') = (0.257 \cos x - 0.857 \cos 3x + 0.0515 \cos 5x)B^2$$

Comparing Equations (55) and (11),

$$b_n(i'') = 0.409\, A_n(i)$$

Since all terms in the integral for $c_n(i'')$ will reduce to the form

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^n x \sin x\, dx, \quad c_n(i'') = 0$$

Employing Equation (54),

$$i'' = (0.257 \cos x - 0.0857 \cos 3x + 0.515 \cos 5x + \frac{(0.409)(36)}{(2)(35)} \cos x$$

$$- \frac{(0.409)(4)}{(2)(9)} \cos x + \frac{(0.409)(36)}{(2)(55)} \cos 5x)\, B^2$$

Let $F_c(x'')$ refer to the case of a cosinusoidal wave at the input of this system when we have a filter or imperfect audio system that introduces a 10% reduction in the third harmonic component and a shift of 30° in phase of the third harmonic. The equation corresponding to Equation (31) becomes

$$F_c(x'') = 2\left[\frac{4A^2}{3\pi} + \frac{(0.787)(8AB)}{15\pi} + \frac{0.257B^2}{2} + \frac{(0.409)(36)B^2}{35\pi}\right]\cos x$$

$$+ 2\left[\frac{4A^2}{15\pi} + \frac{(0.787)(2AB)(36)}{35\pi} - \frac{0.0857B^2}{2} - \frac{(0.409)(4)B^2}{9\pi}\right]\cos 3x$$

$$+ 2\left[\frac{0.0515B^2}{2} - \frac{4A^2}{105\pi} + \frac{(0.787)(2AB)(20)}{63\pi} + \frac{(0.409)(36)B^2}{55\pi}\right]\cos 5x$$

Substituting for A and B from Appendix I,

$$F_c(x'') = 2\left[\frac{4(1.12)^2}{3} - \frac{(0.787)(8)(1.12)(0.160)}{15} + \frac{(0.257)(0.160)^2}{} + \frac{(0.409)(36)(0.160)^2}{35}\right]\cos x$$

$$2\left[\frac{4(1.12)^2}{15} + \frac{(0.787)(2)(1.12)(0.160)(36)}{35} - \frac{(0.087)(0.160)^2}{2} - \frac{(0.160)^2(0.409)(4)}{9}\right]\cos 3x$$

$$2\left[-\frac{4(1.12)^2}{105} - \frac{(0.787)(2)(1.12)(0.160)(20)}{63} + \frac{(0.0515)(0.160)^2}{2} + \frac{(0.409)(36)(0.160)^2}{55}\right]\cos 5x$$

The equation corresponding to Equation (33) is

$$F_c(x'') = 1.027 \cos x + 0.025 \cos 3x + 0.0906 \cos 5x$$

The third harmonic now has an amplitude equal to 2.5% of the fundamental compared to 2.8% of the fundamental component in Equation (33). The fifth harmonic now has an amplitude equal to 8.8% of the fundamental component compared to 9.0% of the fundamental component in Equation (33). So far, the case of a small amount of both amplitude and phase distortion has not seriously altered the output wave. Again, because of the non-linearities within the system, one cannot conclude that these results are necessarily applicable to

the case of speech input. However, the experimental results reported in Chapter IV indicates that the operation of the system with speech input is very similar to the case of a sinusoidal wave input. The small amount of phase and amplitude distortion inherent in the systems reported in Chapter IV did not appreciably affect the quality of the output wave.

No attempt has been made to evaluate the system when there is phase distortion introduced due to differences in path length resulting from multiple ionospheric reflections or interference between ground waves and sky waves. This distortion can be severe. In a practical system it would seem well to consider this and provide a convenient means for eliminating the amplifier performing the inverse function from the system[7].

CHAPTER IV
EXPERIMENTAL RESULTS

In carrying out the experimental work, speech, including certain words particularly selected for the phonemes they contain, was passed through a three kilocycle low pass filter and then recorded on a magnetic tape. This tape was then used as the source of input to the system for all of the tests that followed. This tape also provided a standard of comparison.

The first aim of the experimental work was to demonstrate that such a system could be built . However, this phase already has been reported in Chapter II.

The second aim of the experimental work was to determine the desirability of using a complete system when working with restricted bandwidth rather than one employing only a "power law" compressing amplifier. There is a need to determine this since an amplifier that compresses the speech wave in the manner of the transmitting end amplifier will result in understandable speech without expanding the wave in the receiving end amplifier. Thus there would be no need for the transmitting end amplifier if it did not provide additional advantages.

Tape recordings were made from the output of the entire system and also from the output of the transmitting end amplifier eliminating the receiving end amplifier. The difference in quality of the two outputs indicates a definite improvement when the transmitting end amplifier is employed.

The same process was repeated using a three kilocycle low pass filter at the output of the transmitting end amplifier. This, of course, corresponds to the case where the transmitting bandwidth is to be limited. Use of this filter prevents perfect reproduction primarily due to the phase distortion effects it introduces. Nevertheless, the transmitting end amplifier still gave a very noticeable improvement in the quality of the output. No articulation tests were run to determine the improvement in articulation, but undoubtedly some improvement does result.

In taking the above recordings, it became apparent that the receiving end amplifier can at times give a rather remarkable improvement in signal to noise ratio. The tape recording of the compressed speech wave from the transmitting end amplifier had an annoying amount of noise present on it because of background noises in the shop at the time of recording. In a practical system, the noise present in the compressed speech wave would not arise at the point of transmission but would be in the form of atmospheric noise or interfering signals. When the compressed speech wave was again expanded, the noise dropped to a level where it was barely audible. Obviously, the effect of the receiving end amplifier would be the same regardless of the origin of the noise.

Since the output voltage of the receiving end amplifier is proportional to the square of the input signal, it is apparent that there will be an improvement in signal to noise ratio whenever the signal to noise ratio is greater than one. For instance, if the signal to noise ratio is two to one at the input of the receiving end amplifier,

the signal to noise ratio at the output would be four to one.

In order that there would be some graphical proof of the operation of the system, spectrograms were made by use of a Kay Electric Sona-graph[8]. Photographs of these spectrograms are shown in Figures 19 through 31. In these spectrograms, the speech wave is presented as a three dimensional plot. Frequency is plotted vertically. Time is plotted horizontally, and intensity appears as darkness of the spectrogram.

Figures 19 and 23 show how the words "be" and "pay" appear on this three dimensional plot. These same words were then played into the system from the magnetic tape and spectrograms made at the output. Figures 22 and 26 show how these words looked with no filtering within the system, that is, with no distortion intentionally introduced. Figure 22 should therefore appear the same as Figure 19 and Figure 26 should appear the same as Figure 23. The extent to which this is true is a measure of the excellence of the reproduction. From the figures, it is apparent that the system was reproducing almost perfectly.

Figures 21 and 24 show how these words appeared when a three kilocycle low pass filter was inserted at the output of the transmitting end amplifier with a second three kilocycle low pass filter inserted in the output circuit of the system. Although this introduced distortion that is visible in the spectrograms, it was very difficult to detect by ear.

Figures 20 and 24 show how these words appeared at the output

of the transmitting end amplifier. The distortion in these spectrograms is very apparent. Comparison with Figures 20 and 24 shows the marked improvement in quality that is obtained by using the receiving end amplifier even when employing a restricted bandwidth.

It should be noted that the spectrograms appearing as Figures 21 and 25 were taken with a three kilocycle low pass filter in the output circuit. This filter in general improved the quality of the speech at the output. This effect could have been predicted from the laws of superposition had this been a linear system, as was pointed out in Chapter III. Experimental evidence therefore seems to indicate that results predicted by the laws of superposition may often be approximately correct.

It had been thought possible that since the receiving end amplifier will regenerate some of the harmonics filtered out by a filter at the output of the transmitting end amplifier that in cases of highly restricted bandwidth there might result some improvement in articulation over a linear system. Experimental results indicated the opposite to be true.

An attempt was made to determine the bandwidth requirements of the system versus m for the case of perfect reproduction, that is, complete elimination of filters from within the system. Spectrograms were made from the output of the transmitting end amplifier with various speech sounds at the input and varying values of m. Typical results were obtained with the word, "she". These results are shown in Figures 27 through 31. There was no obvious change in bandwidth requirements over the range in which m could be varied.

In conclusion, it is apparent that this system offers advantages over conventional speech compression or "super modulation" systems by providing a better quality of speech transmission, and in addition it can normally be expected to provide an increase in signal to noise ratio. Unfortunately, no tests have been carried out over long range radio telephone circuits where there is severe phase and amplitude distortion resulting from differences in path lengths between ground and sky waves. It may be that under such circumstances the quality of reproduction would actually be reduced by the transmitting end amplifier. Nevertheless, if the transmitting end amplifier is by-passed at such times, the system will still retain the advantages of conventional speech compression or "super modulation" systems. Thus we can expect this system to normally provide advantages over other systems, and under abnormal condition, its performance will at worst be comparable to other systems.

APPENDIX I

HARMONIC ANALYSIS FOR THE HALF-POWER LAW COMPRESSION OF A COSINUSOIDAL WAVE

This harmonic analysis will provide a description of the wave at the output of the transmitting end amplifier when a sine wave is applied to the input of the system.

This wave may also be described by

$$f(x) = \cos x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = \_\cos x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

A plot of this function is given in Figure 17 of the main text.

A regular periodic function of period T may be represented by a Fourier series as follows:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{n=\infty} \left(A_n \cos 2n\frac{\pi}{T}x + B_n \sin 2n\frac{\pi}{T}x\right)$$

where, when C is any constant

$$A_n = \frac{2}{T}\int_C^{C+T} f(x)\cos\frac{2n\pi x}{T}\,dx$$

$$B_n = \frac{2}{T}\int_C^{C+T} f(x)\sin\frac{2n\pi x}{T}\,dx$$

A plot of this function is given in Figure 14 of the main text.

Taking the region from $\frac{-\pi}{2}$ to $\frac{3\pi}{2}$, $C = \frac{-\pi}{2}$, and $T = 2$

Hence

$$f(x) = \frac{A_o}{2} \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \qquad \text{Eq. (1)}$$

Where

$$A_n = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{3\pi}{2}} f(x) \cos nx\, dx$$

$$\text{Eq. (2)}$$

$$B_n = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{3\pi}{2}} f(x) \sin nx\, dx$$

In the wave form shown in Figure 17 of the main text, $f(x) - f(x + \pi)$, and therefore it will contain no even harmonics. Furthermore, $f(x) = f(-x)$ and hence the Fourier Series consists of cosine terms alone.

Since we are concerned only with cosine terms and odd harmonics, the region from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ is merely the negative of the region from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.

Therefore, we may rewrite Equation (2):

$$A_n = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x) \cos nx dx + \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x + \pi) \cos nx dx \qquad \text{Eq. (3)}$$

Let

$$v = x + \pi$$

Then

$$dv = dx$$

$$\cos n\,(v - \pi) \qquad \cos\,(nv - n\pi)$$

Since we are concerned only with terms for which n is odd,

$$\cos n(v - \pi) = -\cos nv.$$

Substituting into Equation (3),

$$A_n \; \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} -f(v)\cos nv\,dv$$

or

$$A_n \; \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos nx\,dx \qquad \text{Eq. (4)}$$

Since it is of interest to know the average value of the function and since it is helpful in the mathematics that follows, the average value of the function, referred to as C, will first be determined.

$$C = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{1/2} x\,dx$$

Since $f(x) = f(-x)$,

$$C = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos^{1/2} x\,dx \qquad \text{Eq. (5)}$$

Reference to Dwight 854. 1[10] gives

$$\int_{0}^{\frac{\pi}{2}} \cos^{m} x dx = \frac{1}{2} \sqrt{\pi} . \frac{\Gamma(\frac{m}{2} \;\; 1/2)}{\Gamma(\frac{m}{2} \;\; 1)} \text{ , if } m > -1 \qquad \text{Eq. (6)}$$

Thus

$$C = \frac{2}{\pi} \left[1/2 \sqrt{\pi} . \frac{(\frac{1/2}{2} + \frac{1}{2})}{(\frac{1/2}{2} + 1)} \right] = \frac{1}{\sqrt{\pi}} . \frac{\Gamma \frac{3}{4}}{\Gamma \frac{5}{4}}$$

Since

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \qquad \text{Eq. (7)}$$

$$\Gamma(\frac{3}{4} + 1) = \frac{3}{4} \Gamma(\frac{3}{4})$$

$$\Gamma(\frac{3}{4} = \frac{4}{3} \Gamma (1.75)$$

$$\Gamma(\frac{3}{4}) = 1.22 \qquad \text{Eq. (8)}$$

$$\Gamma(\frac{5}{4}) = 0.905 \qquad \text{Eq. (9)}$$

By Equations 8 and 9:

$$C = \frac{1}{\sqrt{\pi}} . \frac{1.22}{0.905} = 0.751 \qquad \text{Eq. (10)}$$

From Equation (4),

$$A_1 = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{1/2} x \cos x dx$$

$$A_1 = \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x dx \qquad \text{Eq. (11)}$$

Reference to Dwight 854.1[10] gives

$$\int_0^{\frac{\pi}{2}} \cos^m x dx = \frac{1}{2}\sqrt{\pi}\,.\ \frac{\Gamma(\frac{m}{2}\ \ \frac{1}{2})}{\Gamma(\frac{m}{2}\ \ 1)}\ , \text{ if } m > -1 \qquad \text{Eq. (6)}$$

Thus

$$C = \frac{2}{\pi}\left[\frac{1}{2}\sqrt{\pi}\,.\ \frac{(\frac{1/2}{2} + \frac{1}{2})}{(\frac{1/2}{2} + 1)}\right] = \frac{1}{\sqrt{\pi}}\,.\ \frac{\Gamma\frac{3}{4}}{\Gamma\frac{5}{4}}$$

Since

$$\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha) \qquad \text{Eq. (7)}$$

$$\Gamma(\frac{3}{4} + 1) = \frac{3}{4}\,\Gamma(\frac{3}{4})$$

$$\Gamma(\frac{3}{4} = \frac{4}{3}\,\Gamma\,(1.75)$$

$$\Gamma(\frac{3}{4}) = 1.22 \qquad \text{Eq. (8)}$$

$$\Gamma(\frac{5}{4}) = 0.905 \qquad \text{Eq. (9)}$$

By Equations 8 and 9:

$$C = \frac{1}{\sqrt{\pi}}\ .\ \frac{1.22}{0.905} = 0.751 \qquad \text{Eq. (10)}$$

From Equation (4),

$$A_1 = \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{1/2} x \cos x dx$$

$$A_1 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x dx \qquad \text{Eq. (11)}$$

Let

$$dv = \cos x \qquad u = \cos^{1/2} x dx$$

$$v = \sin x \qquad du = \frac{1}{2} \cos x^{-1/2} \sin x dx$$

$$A_1 = \frac{2}{\pi}\left[\cos^{1/2} x \sin x \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \frac{2}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{-1/2} x \sin^2 x dx$$

$$= 0 + \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{-1/2} x(1 - \cos^2 x)\, dx$$

$$= \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{-1/2} x dx - \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x dx$$

From Equation (11), and since $f(x) = f(-x)$,

$$\frac{3}{2} A_1 = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos^{-1/2} x dx$$

Employing Equation (6):

$$A_1 = \frac{2}{3\ \pi} \cdot \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \qquad \text{Eq. (12)}$$

By Equation (7)

$$\Gamma(\frac{1}{4} + 1) = \frac{1}{4}\ \Gamma(\frac{1}{4})$$

$$\Gamma(\frac{1}{4}) = 4\ \Gamma(\frac{5}{4})$$

$$= 3.62$$

From Equations (8), (9) and (12),

$$A_1 = \frac{2}{2\pi} \cdot \frac{3.62}{1.22}$$

$$A_1 = 0.112 \qquad \text{Eq. (13)}$$

$$A_3 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{1/2} x \cos 3x dx$$

$$= \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 4\cos^{7/2} x - 3\cos^{3/2} x \qquad \text{Eq. (14)}$$

From Equations (11) and (13):

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x\, dx = (\pi)(0.559) \qquad \text{Eq. (15)}$$

$$A_3 = \frac{8}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x - (6)(0.559)$$

Let

$$u = \cos^{5/2} x \qquad dv = \cos x dx$$

$$du = -\frac{5}{2}\cos^{3/2} x \sin x dx \qquad v = \sin x$$

$$A_3 = \frac{8}{\pi}\left[\cos^{5/2} x \sin x\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \frac{8}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{5}{2}\cos^{3/2} \sin^2 x dx - (3)(0.559)$$

$$A_3 = 0 + \frac{20}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x dx - \frac{20}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x dx - (3)(0.559)$$

Substituting from Equations (14) and (15)

$$\cos^{7/2} x = \frac{\pi}{2}[A_3 + (3)(0.559)]$$

$$A_3 = \frac{20}{\pi}(\pi)(0.559) - \frac{20}{\pi}(\frac{\pi}{4})[A_3 s(0.559)] - (6)(0.559)$$

$$= \frac{4}{7}(0.559)[10 - \frac{15}{2} - 3]$$

$$= 0.160$$

From Equation (4)

$$A_5 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{1/2} x \cos 5 x$$

$$A_5 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (16 \cos^{11/2} x - 20 \cos^{7/2} x + 5 \cos^{3/2} x) dx \quad \text{Eq. (17)}$$

$$= \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 16 \cos^{11/2} x dx - \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (20 \cos^{7/2} x - 5\cos^{3/2} x) dx$$

Let $v = \cos^{9/2} x$, $du = \cos x dx$

Then

$$dv = -\frac{9}{2}\cos^{7/2} x \sin x dx \quad , \quad u = \sin x$$

$$A_5 = \frac{2}{\pi}\left[16 \cos^{9/2} x \sin x \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$+ \frac{2}{\pi}\left[16\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{9}{2} \cos^{7/2} x \sin^2 x dx - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (20 \cos^{7/2} x - 5 \cos^{3/2} x) dx \right]$$

$$= \frac{2}{\pi}\left[16(0) + 16 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{9}{2} (\cos^{7/2} x - \cos^{11/2} x) dx \right.$$

$$\left. - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (20 \cos^{7/2} x - 5 \cos^{3/2} x) dx \right.$$

$$= \frac{2}{\pi}\left[72 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x dx - 72 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{11/2} x dx \right.$$

$$\left. - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (20 \cos^{7/2} x - 5 \cos^{3/2} x) dx \right.$$

From Equation (17)

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{11/2} x dx = \frac{\pi}{2} \left(\frac{1}{16}\right) \left[A_5 + \frac{2}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (20 \cos^{7/2} x - 5 \cos^{3/2} x) dx \right]$$

$$A_5 = \frac{2}{\pi}\left[52 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x dx - \frac{\pi}{2}\left(\frac{9}{2}\right) A_5 - 90 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x dx \right.$$

$$+ \frac{45}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} xdx \;+ 5 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} xdx$$

$$A_5 = \frac{2}{11}\left(\frac{2}{\pi}\right)\left[-58 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} xdx + \frac{55}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} xdx \qquad \text{Eq. (18)}\right.$$

From Equation (14)

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x\,dx = \frac{\pi}{2} \cdot \frac{1}{4} A_3 + \frac{3}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2} x\,dx \qquad \text{Eq. (19)}$$

From Equations (15) and (16)

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x\,dx = \frac{\pi}{2}\left[\frac{1}{4}(-0.160) + \frac{3}{2}(0.559)\right]$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{7/2} x\,dx = \frac{\pi}{2}(0.7985) \qquad \text{Eq. (19)}$$

From equations (15), (18) and (19),

$$A_5 = \frac{2}{11}\left(\frac{2}{\pi}\right)\left[-38\left(\frac{\pi}{2}\right)(0.7985) + 55\left(\frac{\pi}{2}\right)(0.559)\right]$$

$$A_5 = 0.0731$$

This completes the Fourier series up through terms of the fifth order for the waveform in Figure 14 of Chapter III. Hence

$$f(x) = 1.120 \cos x - 0.160 \cos 3x + 0.731 \cos 5x$$

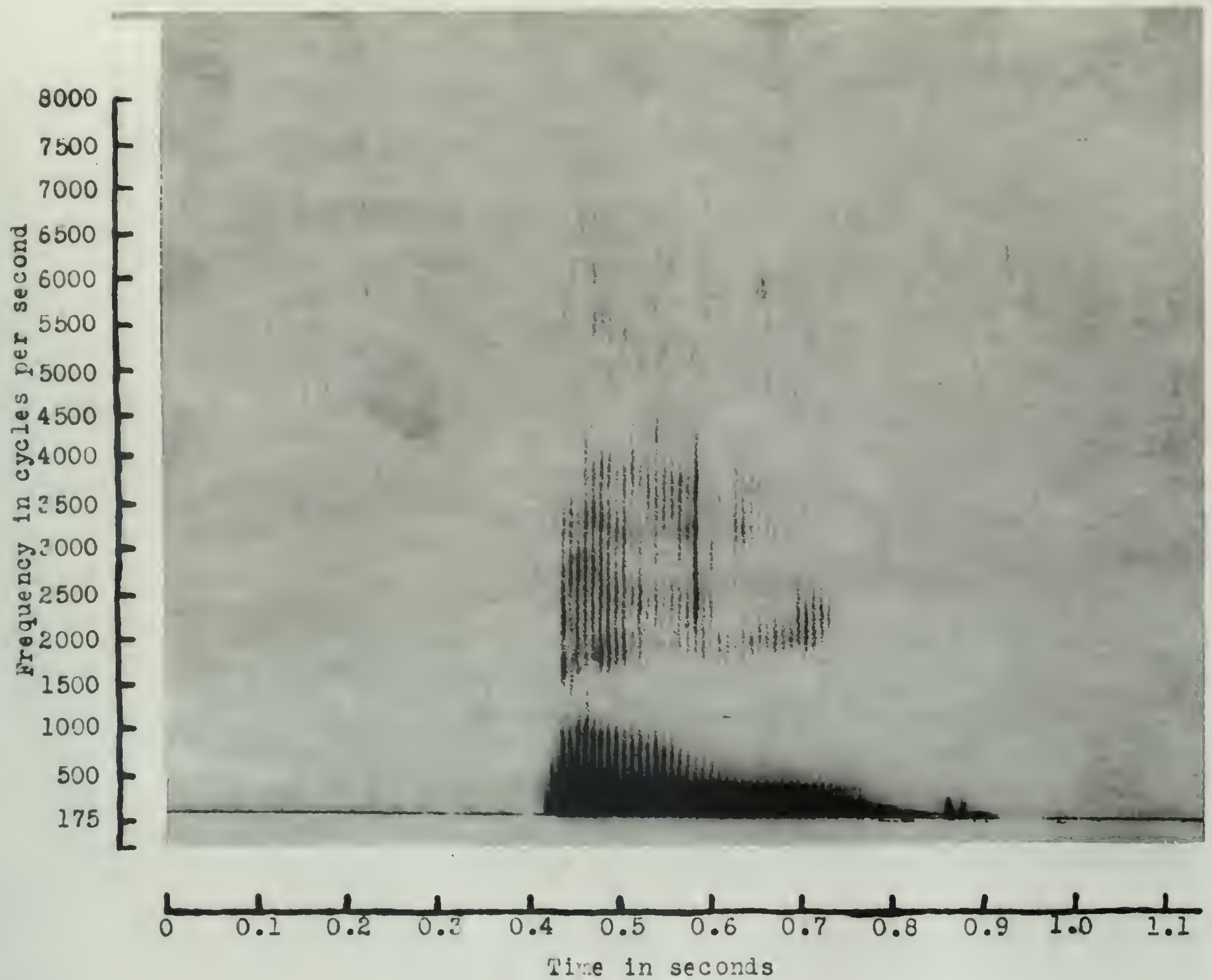


Figure 19

Spectrogram of the word "be" at the input to the system

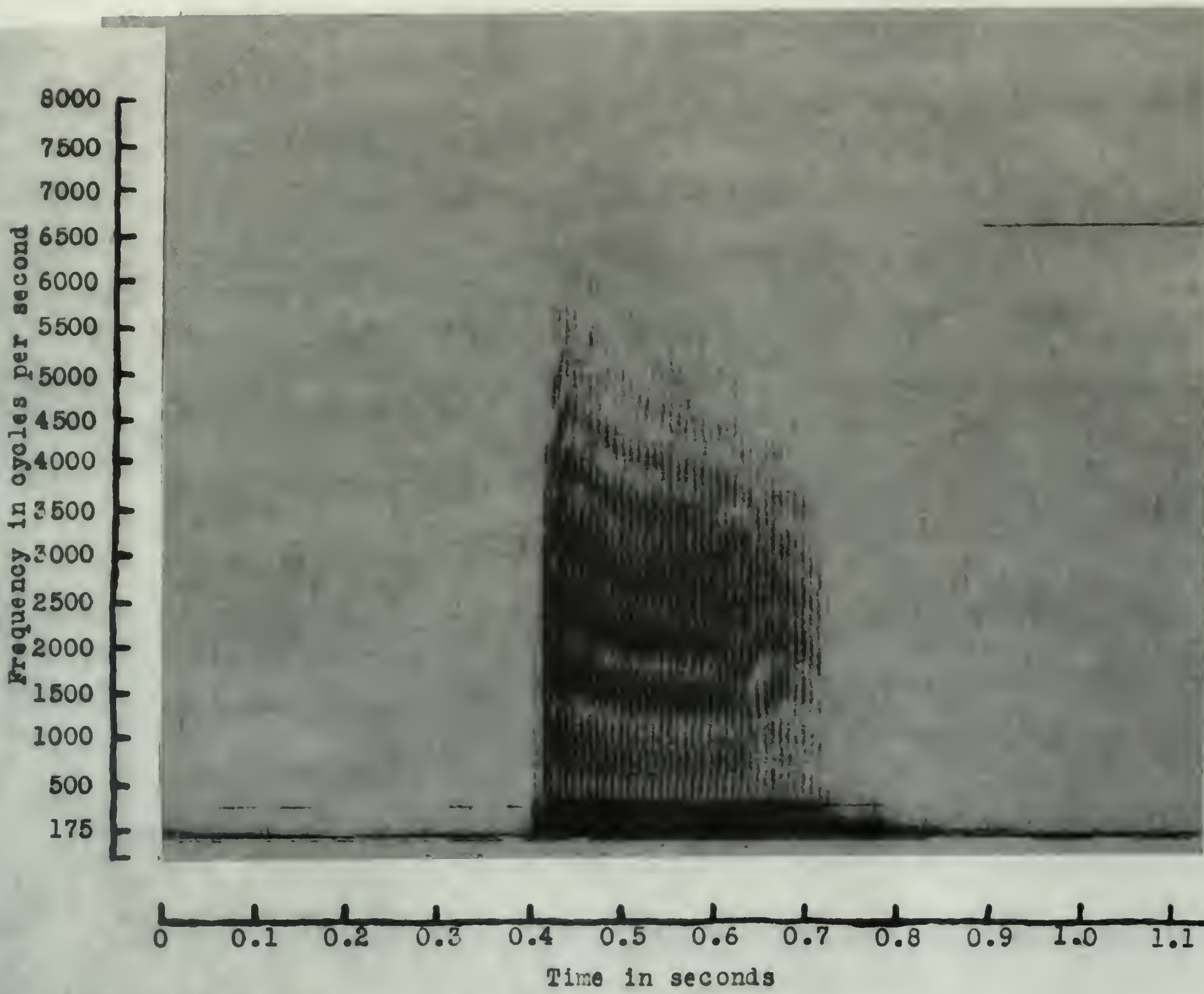


Figure 20

Spectrogram of the word "be" at output of 3,000 cycle low pass filter following transmitting end amplifier

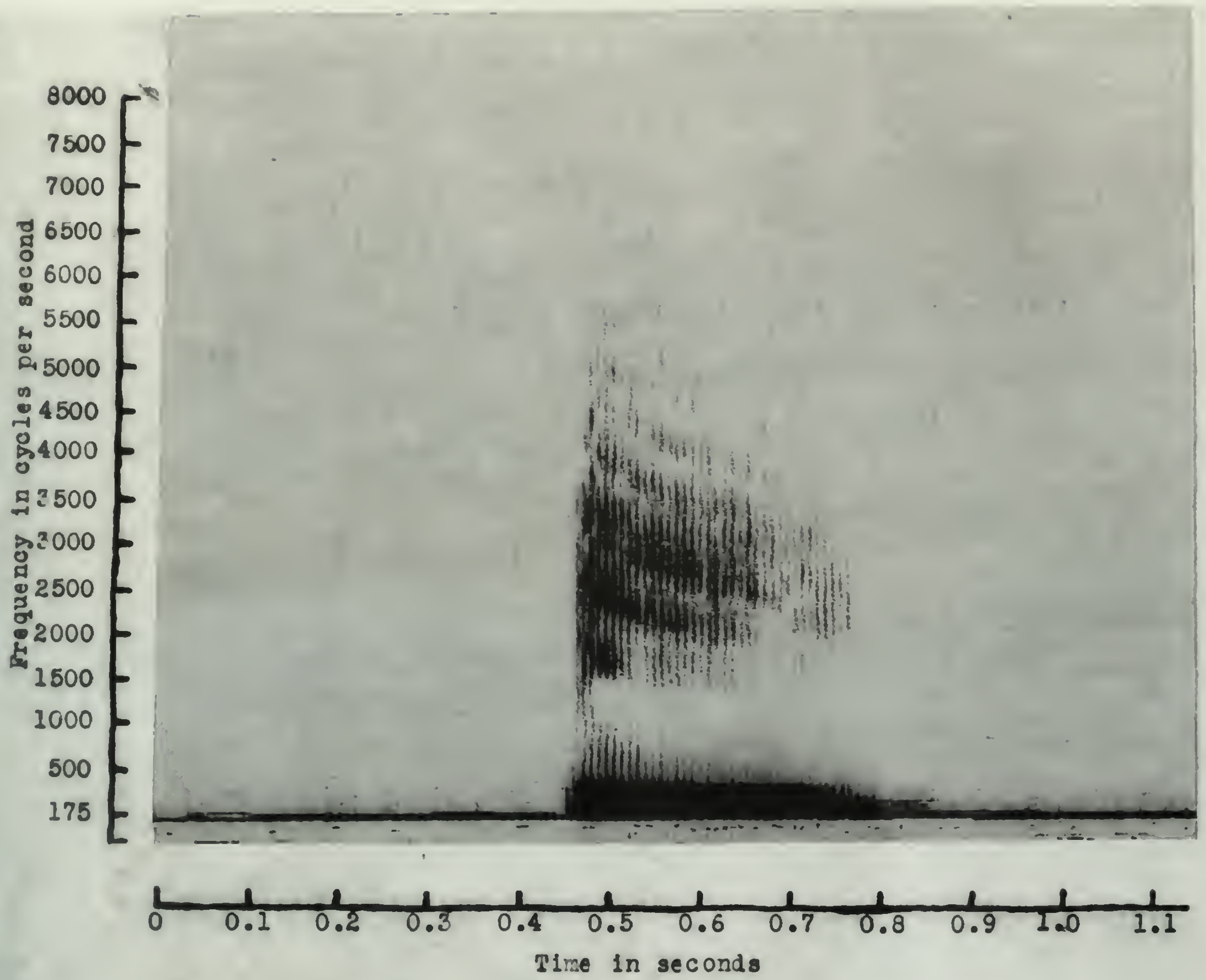


Figure 21

Spectrogram of the word "be" at output of system with a 3.000 cycle low pass filter at output of transmitting end amplifier and also in the output circuit of the system.

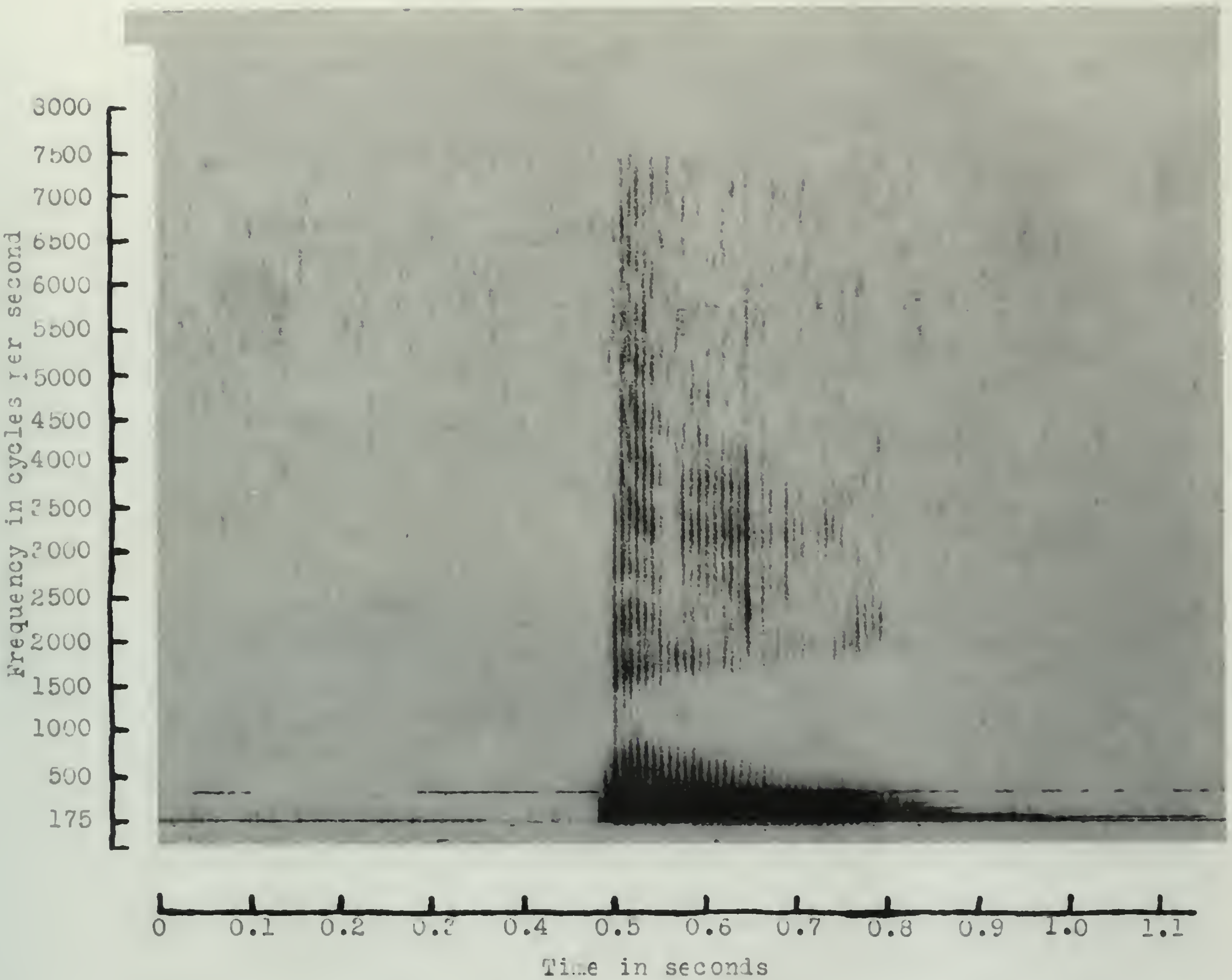


Figure 22

Spectrogram of the word "be" at output of the system with no filters in the system.

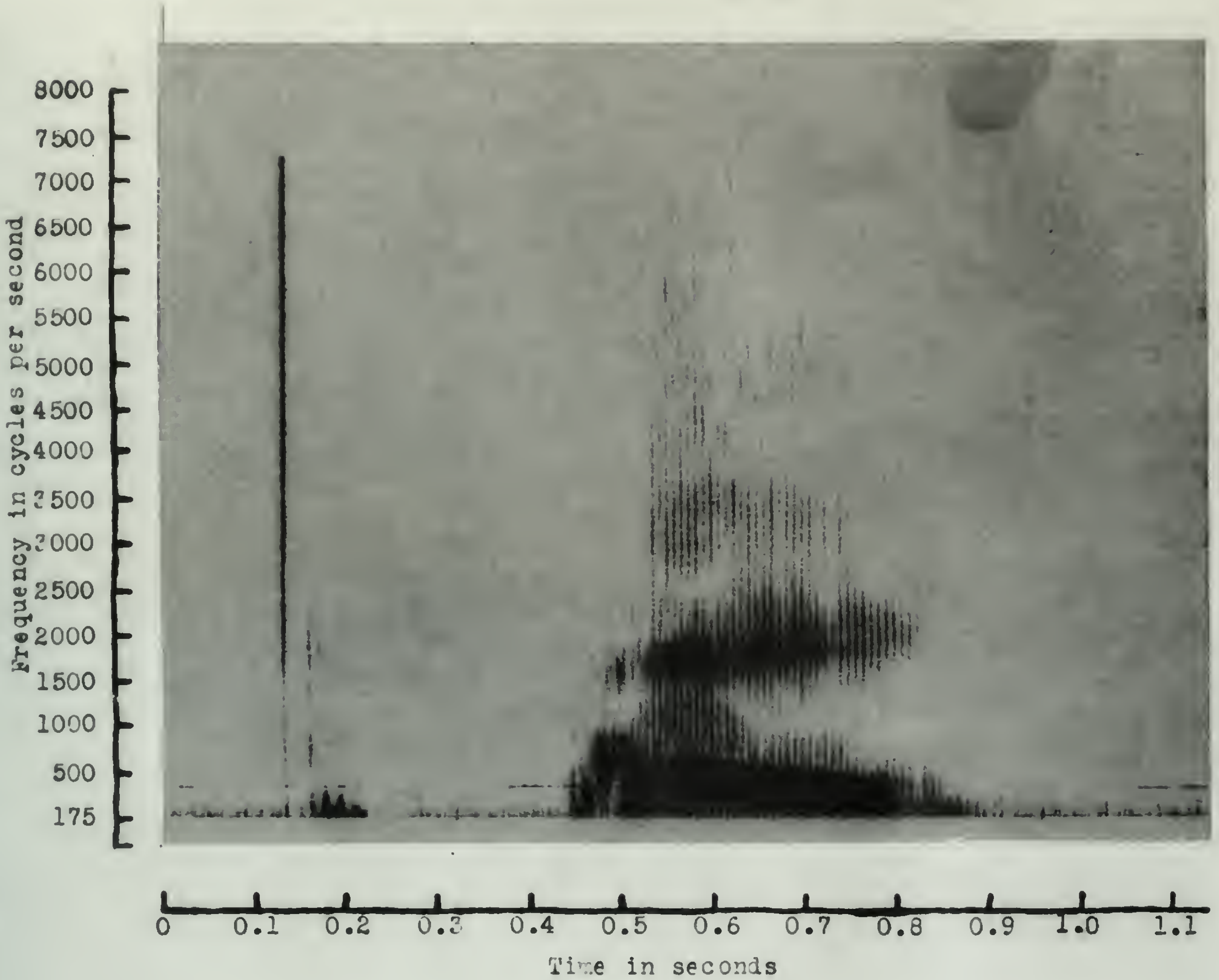


Figure 23

Spectrogram of the word "pay" at the input to the system.

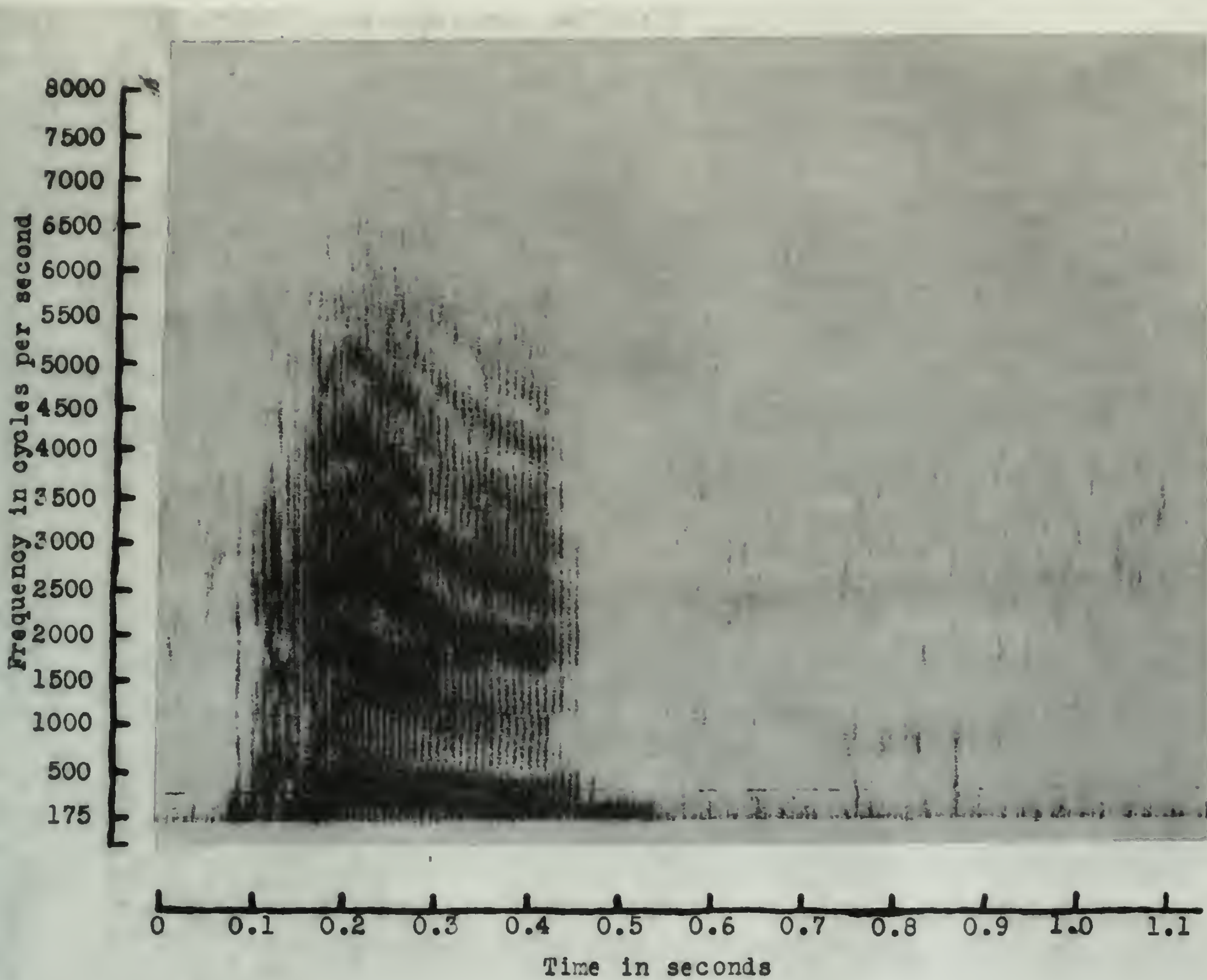


Figure 24.

Spectrogram of the word "pay" at output of 3,000 cycle low pass filter following transmitting end amplifier.

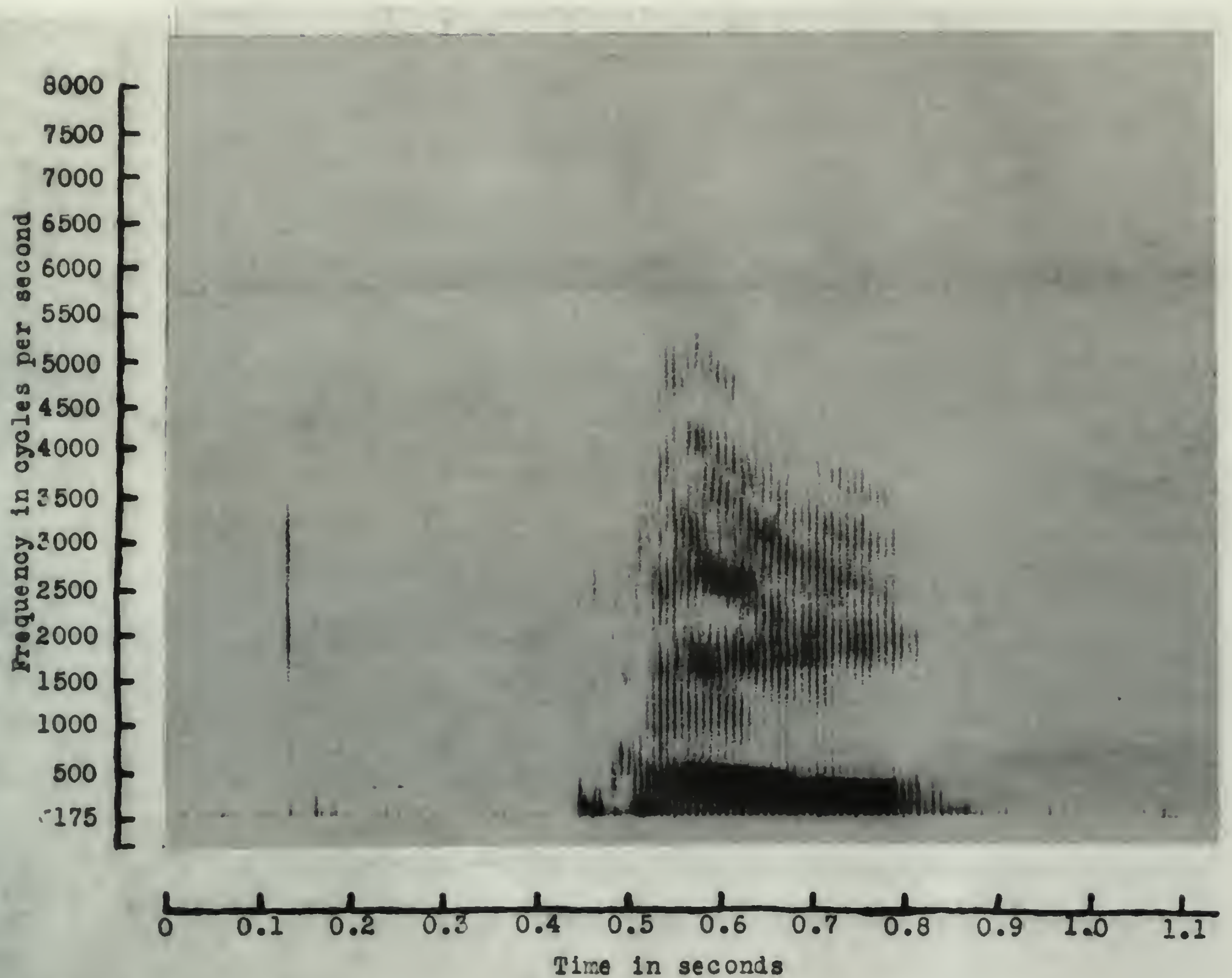


Figure 25

Spectrogram of the word "pay" at output of system with a 3,000 cycle low pass filter at output of transmitting end amplifier and also in the output circuit of the system.

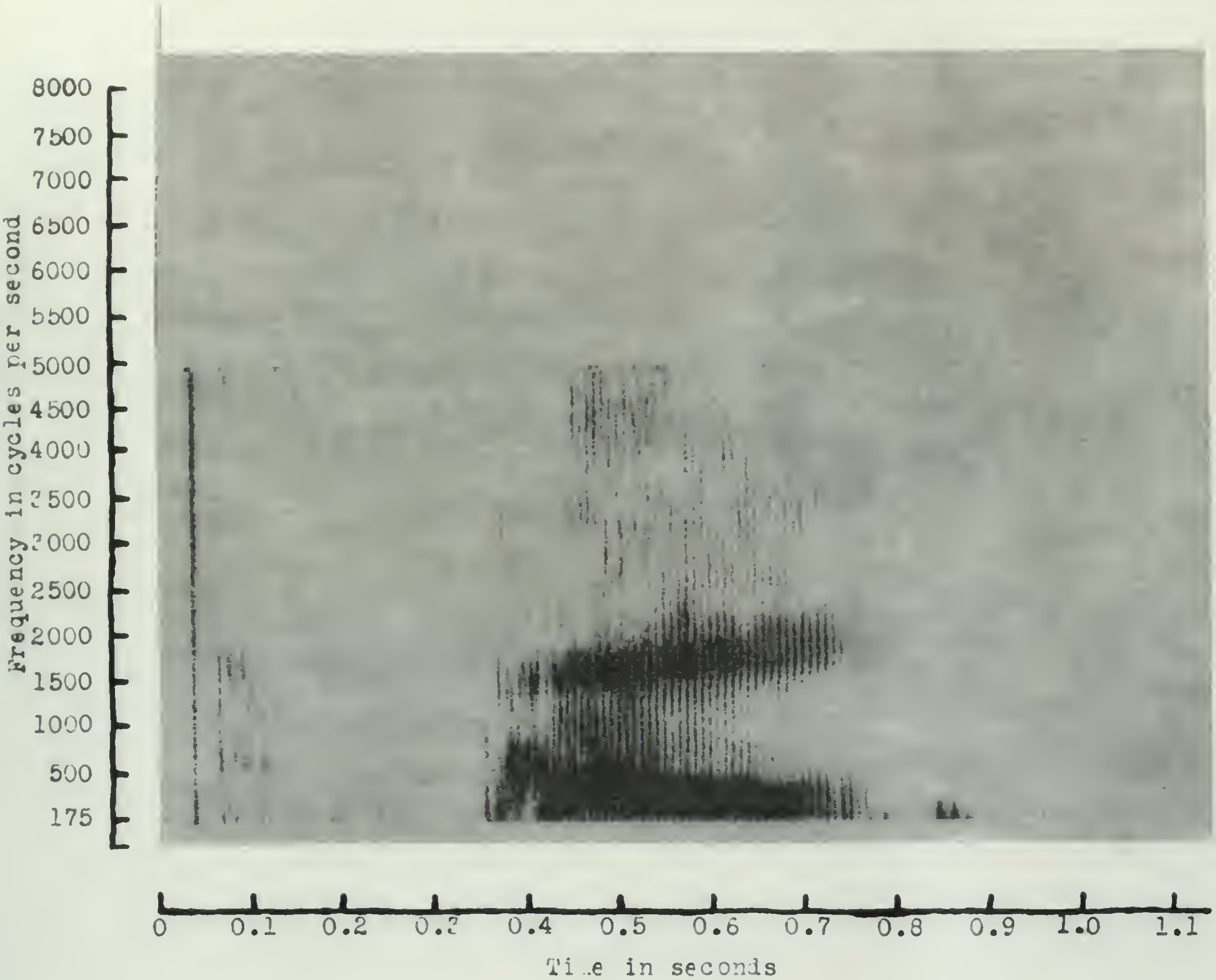


Figure 26

Spectrogram of the word "pay" at output of the system with no filters in the system.

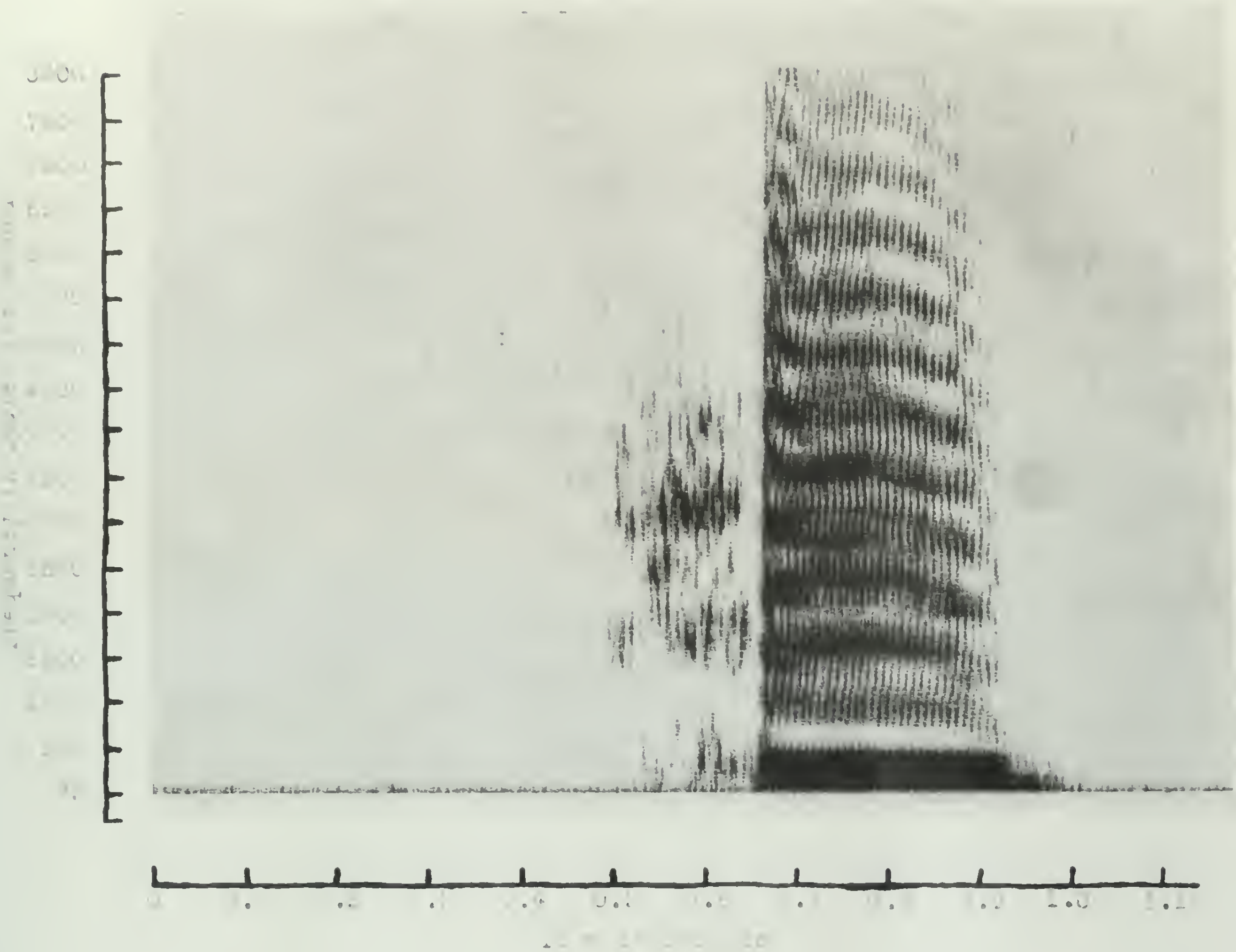


Figure 27

Spectrogram of the word "she" at output of the transmitting end amplifier with m = 0.375.

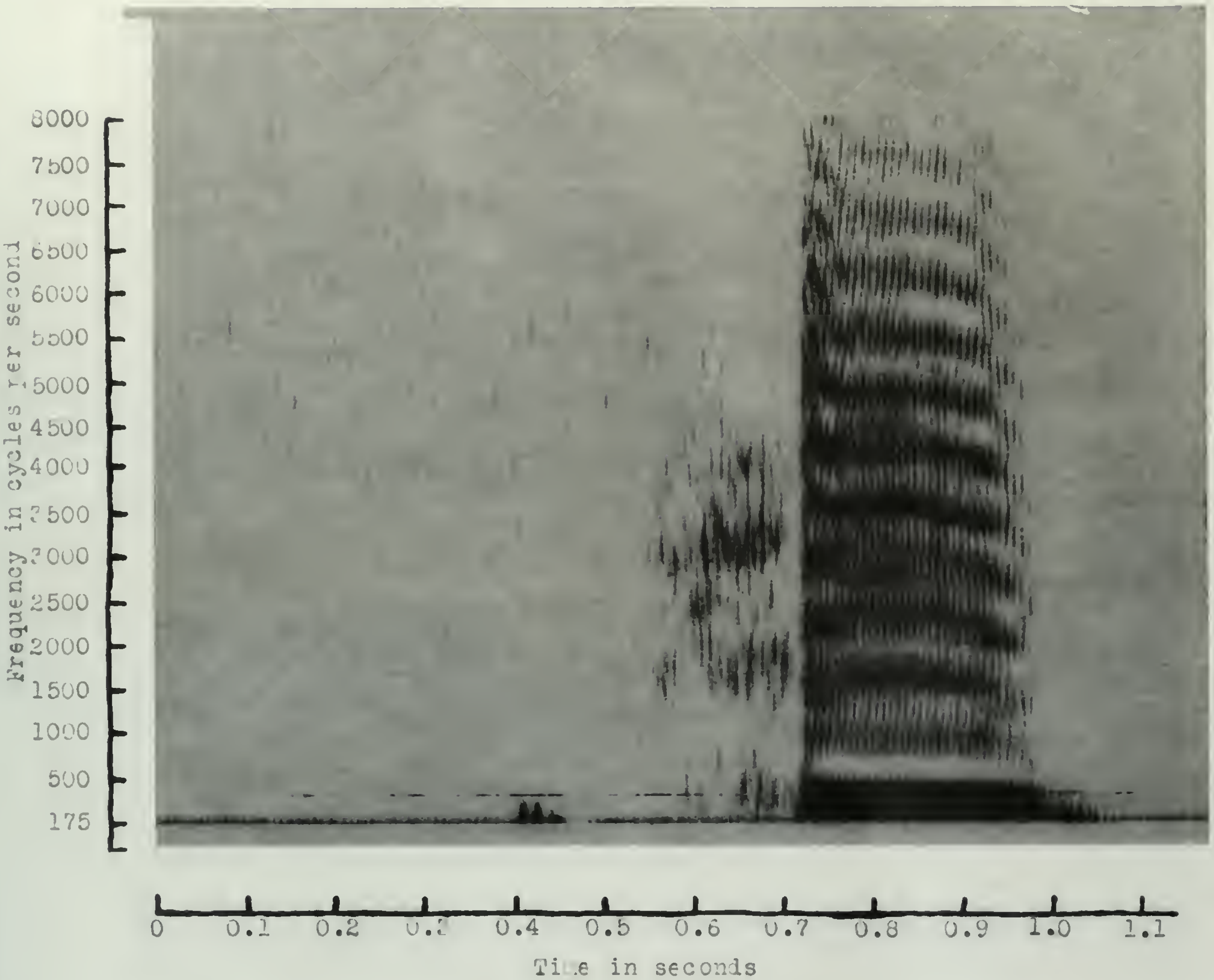


Figure 28

Spectrogram of the word "she" at output of the transmitting end amplifier with m = 0.425

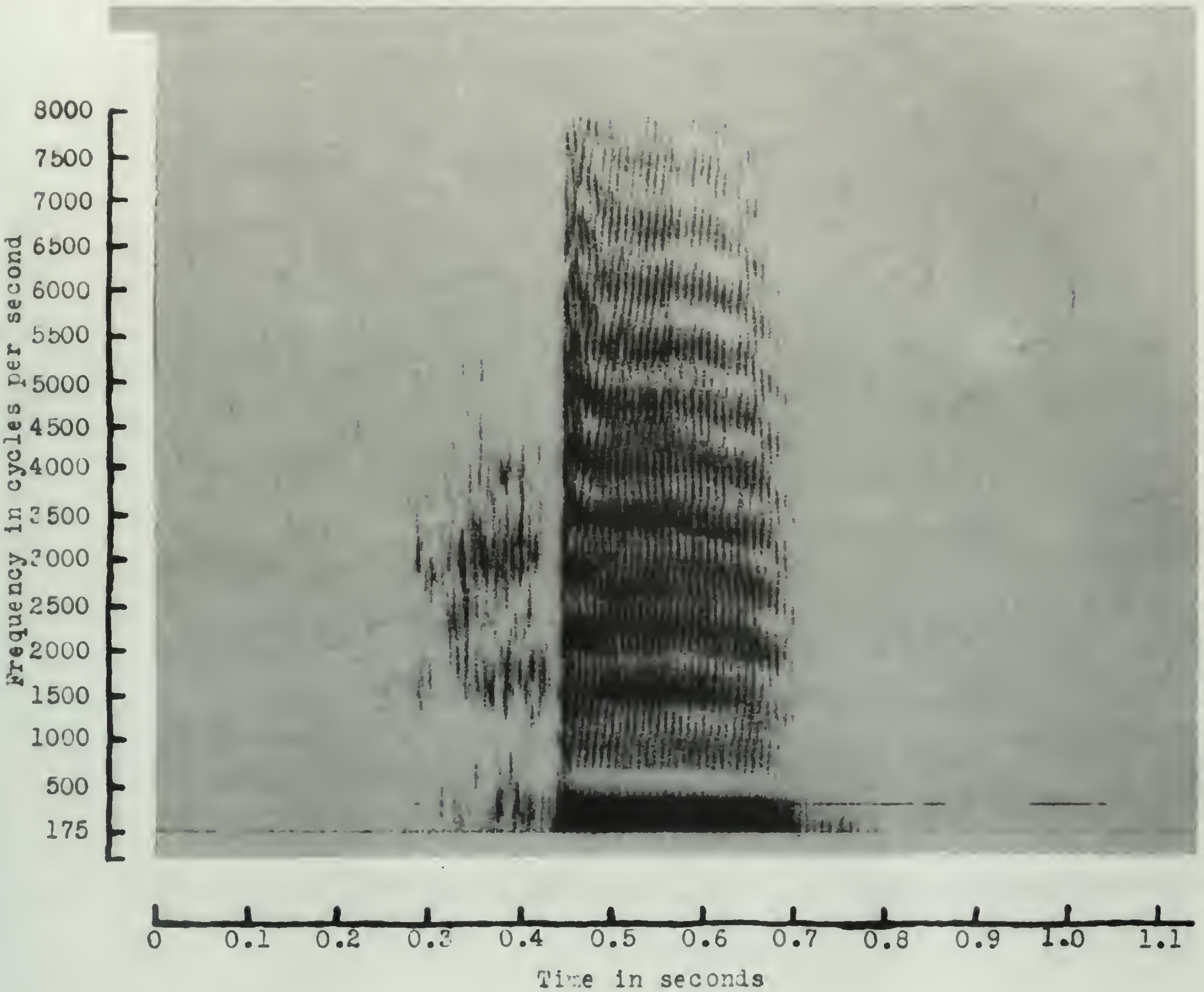


Figure 29

Spectrogram of the word "she" at output of the transmitting end amplifier with m = 0.475

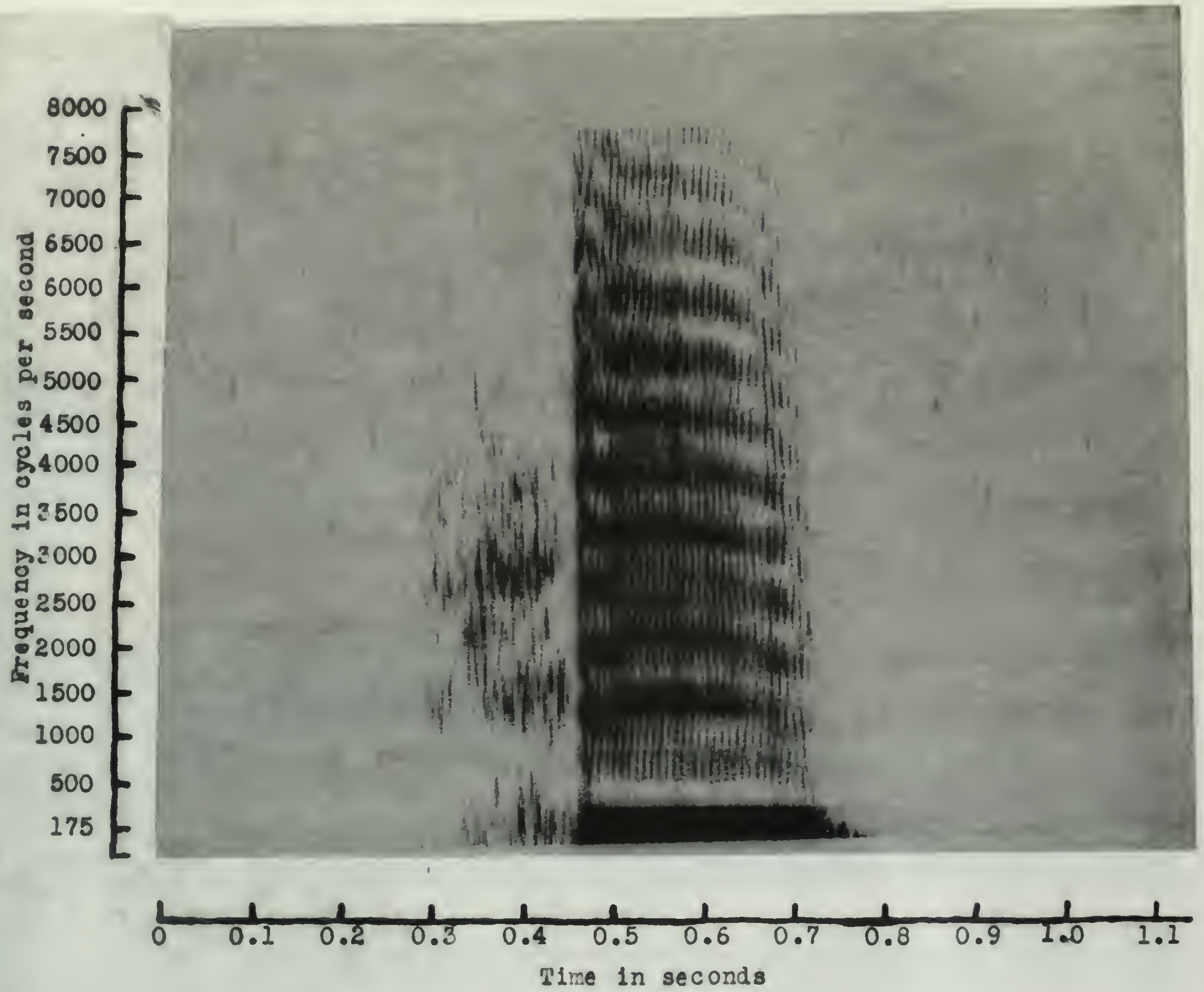


Figure 30

Spectrogram of the word "she" at output of the transmitting end amplifier with m = 0.525

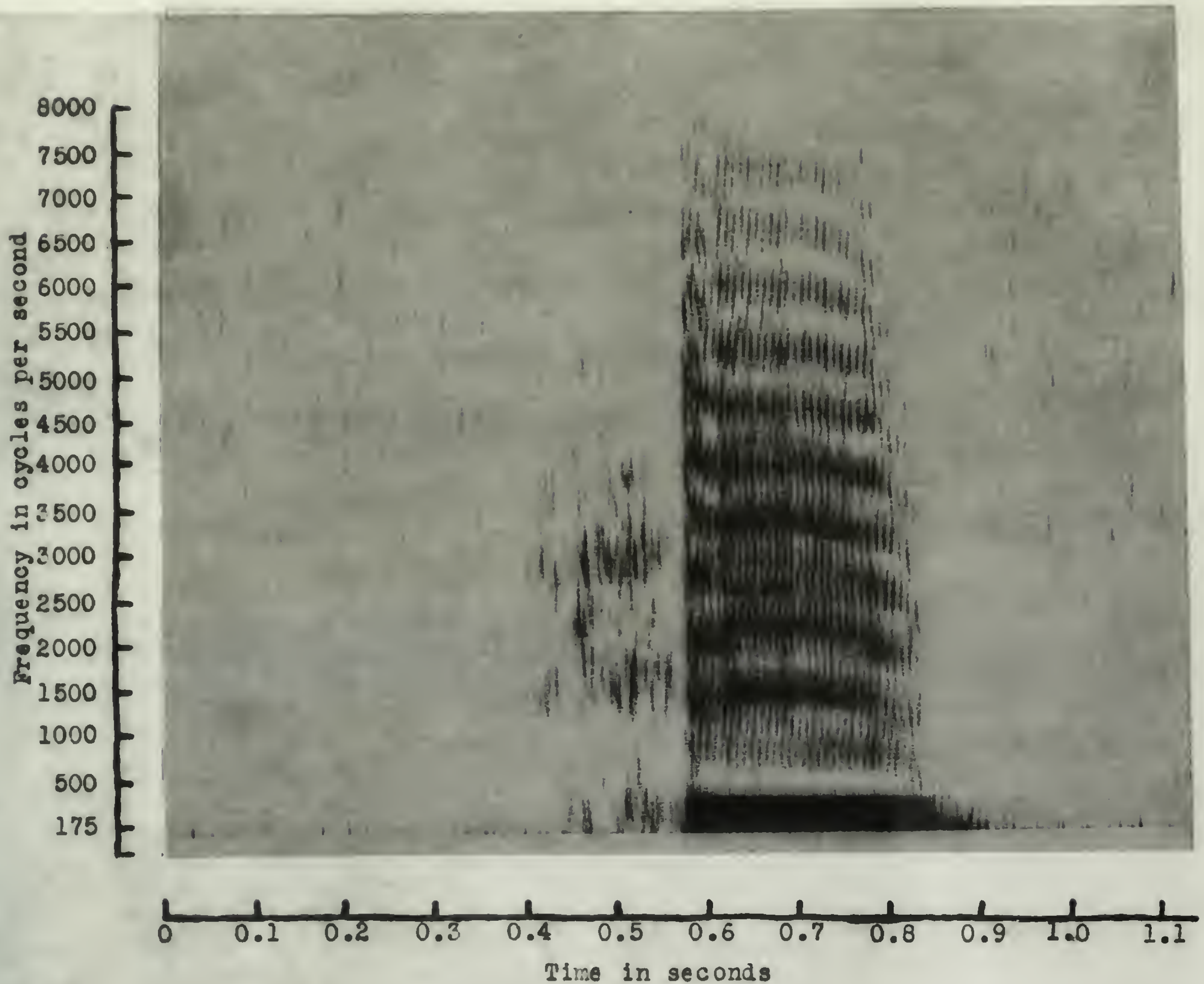


Figure 31.

Spectrogram of the word "she" at output of the transmitting end amplifier with m = 0.546.

BIBLIOGRAPHY

1. Beranek, L. L. Acoustics Measurements, J. Wiley and Sons, 1949, p. 391.

2. Straus, Oliver H., The Relation of Phonetics and Linguistics to Communication Theory, The Journal of the Acoustical Society of America, Volume 22, Number 6, November, 1950.

3. Potter, R. K., Kopp, G. A., and Green, H. C., Visible Speech, D. Van Nostrand Co., 1947, Chapter 5.

4. Fletcher, Harvey, Speech and Hearing, D. Van Nostrand Co. 1929, Chapter 5.

5. McCann, D. G., Wilts, C. H., and Locanthi, B. N., Electronic Techniques Applied to Analogue Methods of Computation, Proceedings of the Institute of Radio Engineers, Volume 37, Number 8, August 1949.

6. Tucker, D. G., Rectifier Resistance Laws, Wireless Engineer, April, 1948, p. 117.

7. Brown, R., Martin, De Loss K., and Potter, Ralph K., Some Studies in Radio Broadcast Transmission, Proceedings of the Institute of Radio Engineers, February, 1929, pp. 57-132.

8. Gruenz, O. A., An 8000 Cycle Sound Spectrograph, Bell Laboratories Record, June, 1951, pp. 256-261.